Biconvex reformulation: A new safe constraint reformulation

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Background

$$\inf_{(x^i(t),x^j(t))} \|x^i(t) - x^j(t)\|_2 \geq d_{min}$$
$$\text{s.t. } f^l(x^i(t)) \leq 0,\ l = 1, 2, \ldots, p$$
$$g^l(x^j(t)) \leq 0,\ l = 1, 2, \ldots, p$$

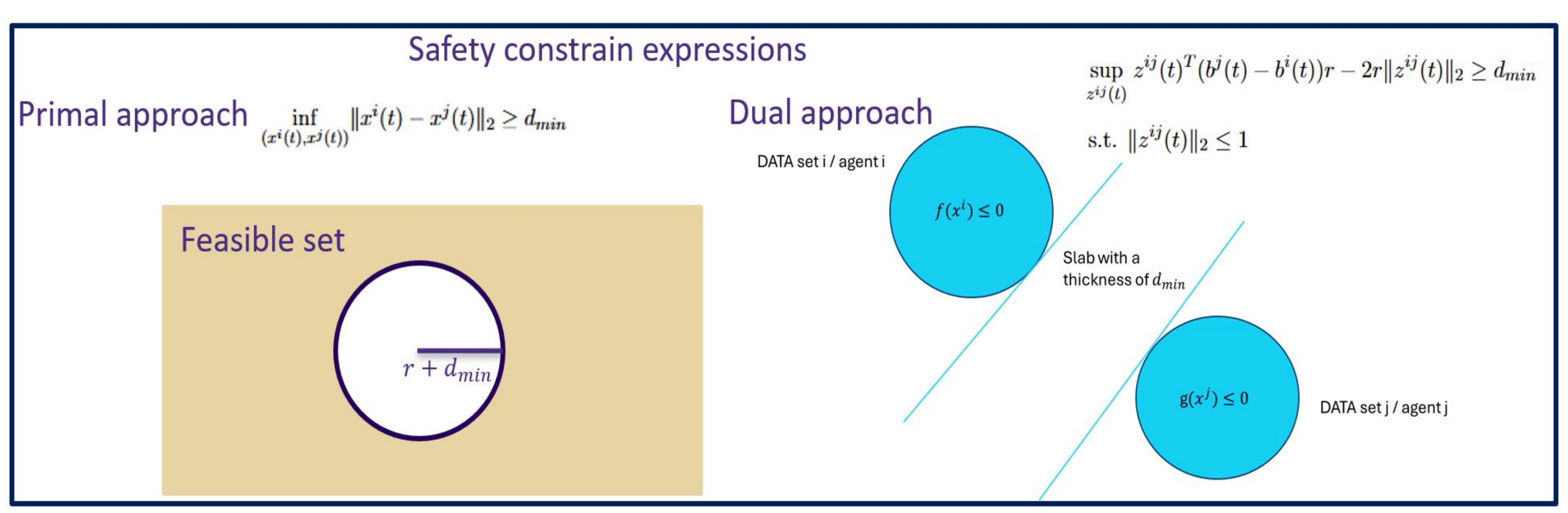
- Traditional pure linear approximation approach are unable to represent the safety constraint accurately
- The need to carefully tune the slack variable penalty makes it more sensitive to the parameter choosing, and this sometimes make the solution hard to converge to a good or even feasible solution when complex dynamics and constraints are considered

$$L(\mathbf{d}_k^i, \mathbf{w}_k^i, \mathbf{s}_k^{\prime ij}, \mathbf{v}_k^i) = C(\mathbf{x}_k^i + \mathbf{d}_k^i, \mathbf{u}_k^i + \mathbf{w}_k^i) + \sum_{t=0}^{T-1} \lambda \|v_k^i(t)\|_1 + \boxed{\sum_{j=1, j\neq i}^{N} \lambda \max(0, s_k^{\prime ij}(t))}$$

Not hard constrain
Sensitive to λ

$$s_k^{\prime ij}(t) = s_k^{ij}(t) + \frac{\partial s_k^{ij}(t)}{\partial x^i(t)}\Big|_{x_k^i(t)} d_k^i(t)$$

New way to formulate the safety constraint



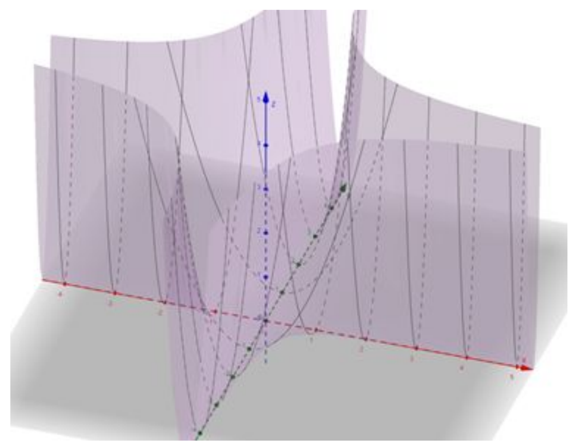
- We are considering the dual problem of the minimum distance problem
- Instead of creating a hole in the solution set as in the regular approaches, the dual approach resembles the data classification problem in which agents (or data sets) are separated by the supporting hyper slab with some thickness
- Produce the biconvex coupled constrains

Biconvex problem

$$\sup_{z^{ij}(t)} z^{ij}(t)^T(b^j(t) - b^i(t))r - 2r\|z^{ij}(t)\|_2 \geq d_{min}$$
$$\text{s.t. } \|z^{ij}(t)\|_2 \leq 1$$

- The reformulated safety constraint becomes a biconvex constraint

A biconvex function
$a(x, y) = (x\,y)^2$



- This reformulation yields a biconvex problem that can be split into two convex subproblems

Alternating Convex Searching (ACS)

Primal minimization (Upper level)

$$\min_{\mathbf{d}_k^i, \mathbf{w}_k^i, \mathbf{v}_k^i} L(\mathbf{d}_k^i, \mathbf{w}_k^i, \mathbf{v}_k^i) + \sum_{j=1, j\neq i}^{N} \lambda \max(0, s_k^{ij}(t))$$
$$\text{s.t. } x_k^i(t+1) + d_k^i(t+1) = f(x_k^i(t), u_k^i(t)) + A_d^i(t) d_k^i(t) + B_d^i(t) w_k^i(t) + I^{n\times n} v_k^i(t),\ \forall t = 0, \ldots, T-1$$
$$x^i(0) + d_k^i(0) = \bar{x}^i(0)$$
$$x^i(T) + d_k^i(0) = \bar{x}^i(T)$$
$$s_k^{ij}(t) = -Q(z^{ij}(t), x_k^i(t)), \forall t = 0, \ldots, T$$
$$s_k^{ij}(t) \leq 0, \forall t = 0, \ldots, T$$

Dual update (Lower level)

$$\min_{(\mathbf{z}_k^i)} \sum_{j=1, j\neq i}^{N} \lambda \max(0, s_k^{ij}(t))$$
$$\text{s.t. } s_k^{ij}(t) = -Q(z^{ij}(t), x_k^i(t))$$
$$\|z^{ij}(t)\|_2 \leq 1$$

- ACS allows biconvex problems to solved in a bilevel fashion with strong convergence to partial minimum

$$\min_{x,y} f(x, y)$$
$$\text{s.t. } axy + b = 0$$
$$x \in X \subseteq \mathbb{R}, y \in Y \subseteq \mathbb{R}$$

$$\min_{x_{k+1}} f(x_{k+1}, y_k) \quad \text{Upper level}$$
$$\text{s.t. } ax_{k+1}y_k + b = 0$$
$$x_{k+1} \in X \subseteq \mathbb{R}$$

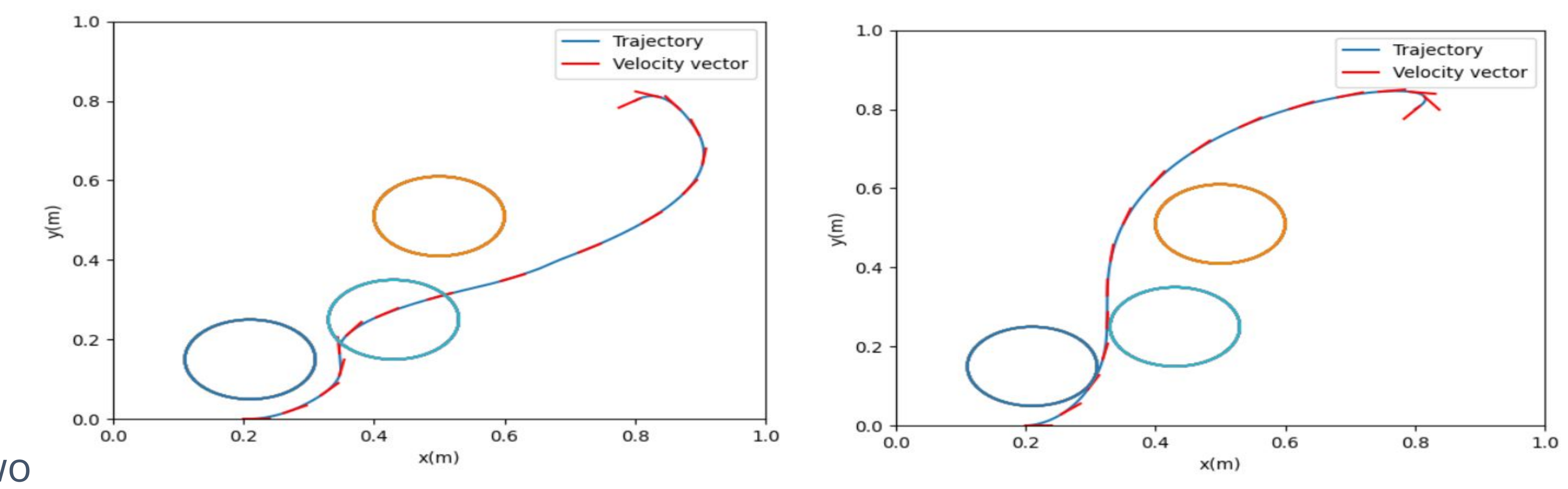
$$\min_{y_{k+1}} f(x_{k+1}, y_{k+1}) \quad \text{Lower level}$$
$$\text{s.t. } ax_{k+1}y_{k+1} + b = 0$$
$$y_{k+1} \in Y \subseteq \mathbb{R}$$

$$f(x^*, y^*) \leq f(x, y^*),\ \forall x \in B_{y^*}$$
$$f(x^*, y^*) \leq f(x^*, y),\ \forall y \in B_{x^*}$$

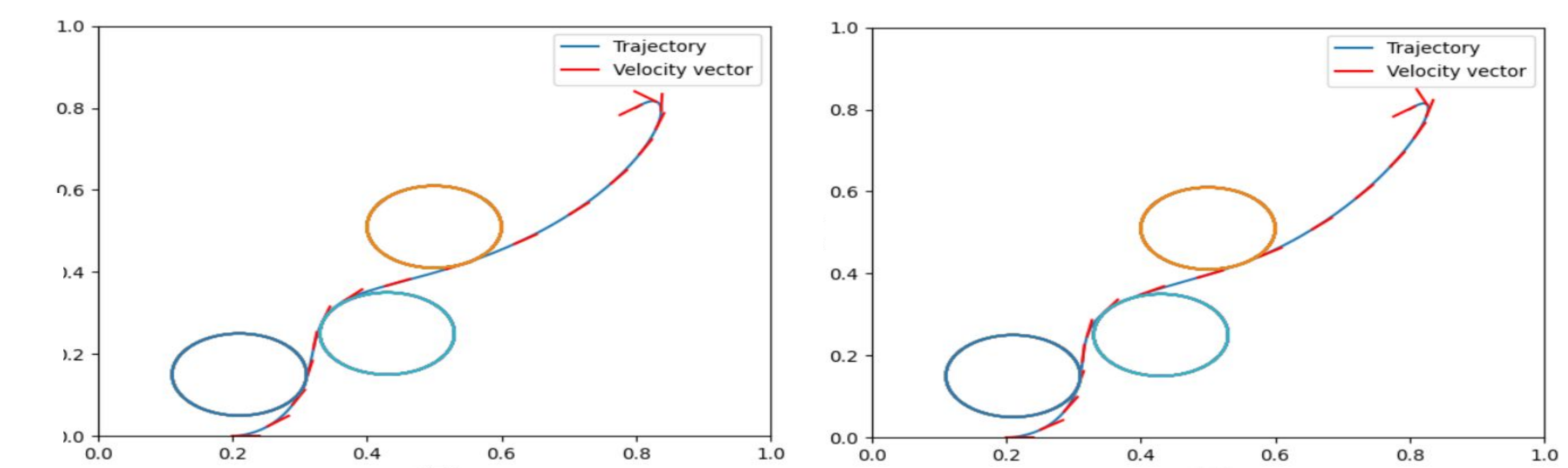
Simulation results

- Much faster convergence per iteration compared to the traditional linearization approaches
- Strict feasibility for the safe constraint

Solutions after one and ten iterations with pure linearization

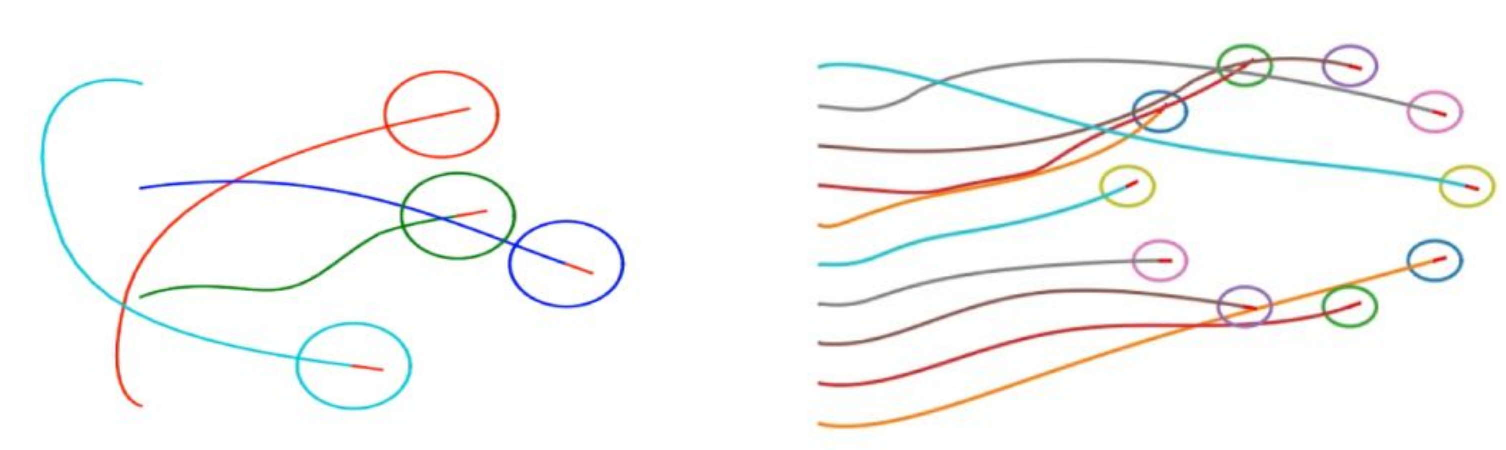


Solutions after one and ten iterations with biconvex reformulation



Iteration to converge (SVM)	2
Iteration to converge (Pure linearization)	8
Total control $\|u\|_2$ (SVM)	0.02095 m/s^2
Total control $\|u\|_2$ (Pure linearization)	0.02135 m/s^2

Multiagent scenario



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