Comparing Reduced Fluid Models to the Full Two-Fluid Plasma System

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Abstract

The two-fluid plasma model is studied and compared to reduced fluid models such as Hall-MHD. Three asymptotic approximations are independently applied to the full two-fluid plasma model to obtain the reduced models which include charge neutrality, infinite speed of light and negligible electron inertia. Applying all three approximations gives Hall-MHD. Hall-MHD is pursued because it captures additional physics as compared to ideal MHD. The additional physics takes into account two-fluid effects by using the Hall and the diamagnetic drift terms believed to be important in Hall accelerators, Z-pinches, field-reversed configurations, and other such applications. Two-fluid effects become significant when the characteristic spatial scales are small compared to the ion skin depth and the characteristic time scales are short compared to the inverse ion cyclotron frequency. Simulations of electromagnetic plasma shock are performed and the results are compared between the models.
Motivation

• To study the two-fluid plasma model

• To compare Hall-MHD to the two-fluid model to study the physics that is lost or captured by applying the approximations in addition to comparing the simplicity of implementation of the models

• To study and implement a finite electron mass reduced fluid model where the approximations applied to the two-fluid plasma model are $c \rightarrow \infty$ and $n_i = n_e$
Two-Fluid Plasma Model

Euler equations for ion and electron fluids in 1D for simplicity

\[
\frac{\partial}{\partial t} \begin{pmatrix} n_i \\ n_i u_i \\ n_i v_i \\ n_i w_i \\ e_i \\ n_e \\ M n_e u_e \\ M n_e v_e \\ M n_e w_e \\ e_e \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} n_i u_i \\ n_i u_i^2 + p_i \\ n_i u_i v_i \\ n_i u_i w_i \\ (e_i + p_i) u_i \\ n_e u_e \\ M n_e u_e^2 + p_e \\ M n_e u_e v_e \\ M n_e u_e w_e \\ (e_e + p_e) u_e \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{n_i q_i}{m_i} (E_x + v_i B_z - w_i B_y) \\ \frac{n_i q_i}{m_i} (E_y + w_i B_x - u_i B_z) \\ \frac{n_i q_i}{m_i} (E_z + u_i B_y - v_i B_x) \\ n_i q_i (E_x u_i + E_y v_i + E_z w_i) \\ 0 \\ -\frac{n_e q_e}{m_i} (E_x + v_e B_z - w_e B_y) \\ -\frac{n_e q_e}{m_i} (E_y + w_e B_x - u_e B_z) \\ -\frac{n_e q_e}{m_i} (E_z + u_e B_y - v_e B_x) \\ -n_e q_e (E_x u_e + E_y v_e + E_z w_e) \end{pmatrix}
\]
The electromagnetic fields are described by Maxwell equations:

\[
\begin{align*}
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0, \\
\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} &= -\frac{1}{\varepsilon_0} \sum_s q_s \rho_s \mathbf{v}_s \\
\nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0} (q_i n_i + q_e n_e) \\
\nabla \cdot \mathbf{B} &= 0.
\end{align*}
\]

Writing Maxwell’s equations in balance law form in 1D:

\[
\begin{pmatrix}
\frac{\partial}{\partial t} \\
\frac{\partial}{\partial x}
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z \\
B_x \\
B_y \\
B_z
\end{pmatrix}
+ \begin{pmatrix}
0 \\
c^2 B_z \\
-c^2 B_y \\
-\varepsilon_0 (n_i q_i u_i - n_e q_e u_e) \\
-\varepsilon_0 (n_i q_i v_i - n_e q_e v_e) \\
-\varepsilon_0 (n_i q_i w_i - n_e q_e w_e)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
The ion and electron energies $e_i$ and $e_e$ are

$$e_i = \frac{p_i}{\gamma - 1} + \frac{1}{2} n_i (u_i^2 + v_i^2 + w_i^2)$$

$$e_e = \frac{p_e}{\gamma - 1} + \frac{1}{2} M n_e (u_e^2 + v_e^2 + w_e^2)$$

$M$: electron-to-ion mass ratio

$n$: number density

$u, v, w$: velocity in x-, y-, z-directions

$p$: pressure

$E$: electric field

$B$: magnetic field

$q_i, q_e$: ion and electron charges

$c$: speed of light
Asymptotic Approximations to be Applied to the Two-Fluid Model

The approximations applied to the two-fluid model include:

- Charge neutrality, \( n_i = n_e \)

- No displacement currents, \( \varepsilon_0 \to 0, c \to \infty \)

- Ignore electron inertia, \( M \to 0 \)

Applying all 3 of the above together to the two-fluid plasma model gives Hall-MHD

A finite electron mass model is explored by only applying the first 2 approximations above to the two-fluid plasma model.
Hall-MHD Equation System

Written as ideal MHD + source terms:

\[
\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i u_i) = 0
\]

\[
\frac{\partial \rho_i u_i}{\partial t} + \nabla \cdot (\rho_i u_i u_i + P_i \bar{I} - \frac{BB}{\mu_0} + \frac{B^2}{2\mu_0} \bar{I}) = 0
\]

\[
\frac{\partial \varepsilon_i}{\partial t} + \nabla \cdot \left[ \left( \varepsilon_i + P_i + \frac{B^2}{2\mu_0} \right) u_i - \frac{B \cdot u_i}{\mu_0} B \right] = - \frac{1}{\mu_0 n_i q_i^2} \nabla \left( \frac{B^2}{n_i} \right) \cdot (\nabla \times B)
\]

\[
\frac{\partial B}{\partial t} + \nabla \cdot (u_i B - B u_i) = - \nabla \times \left[ \frac{(\nabla \times B) \times B}{n_i q_i \mu_0} - \frac{\nabla P_e}{n_i q_i \mu_0} \right]
\]

The induction equation source term is tough to handle with shock problems such as the BrioWu shock. Need the electron pressure term to be present and/or need to add artificial resistivity to the induction equation to obtain a stable solution.

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Finite Electron Mass Hall-MHD Equation System

- Euler equations for ion fluid with electro-magnetic Lorentz force sources
- Euler equations for electron fluid with electro-magnetic Lorentz force sources
- \( \frac{\partial B}{\partial t} + \nabla \times E = 0 \)
- Setting \( c \to \infty \) gives \( \nabla \times B = \mu_0 nq (u_i - u_e) \)

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Parallel Dispersion Relations for the Two-Fluid Model

Parallel propagation dispersion diagrams, $\omega$ vs $k$

Left: R- and L-mode waves for the two-fluid plasma model

Right: Whistler wave for the two-fluid plasma model that asymptotes at the electron cyclotron frequency

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Parallel propagation dispersion diagram, $\omega$ vs $k$

The whistler wave (blue) grows quadratically without bound for Hall-MHD when skin depth, $\delta_i$ and $\omega_{ci}$ are significant and Hall effects become important. If $\frac{\lambda}{\delta_i} \gg 1$ then the dispersion diagram sits near the origin and resembles ideal MHD.

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Parallel Dispersion Relations for the Finite Electron Mass Hall-MHD Model, $c \to \infty$, $n_i = n_e$

Parallel propagation dispersion diagrams, $\omega$ vs $k$

Left: Wave that asymptotes at the ion cyclotron frequency similar to one of the Hall-MHD waves previously shown

Right: Whistler wave that asymptotes at the electron cyclotron frequency similar to the two-fluid model

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Comparing the Whistler Wave from the Two-Fluid Model, Hall-MHD and the Finite Electron Mass Hall-MHD Model

For the two-fluid (green) and the finite electron mass Hall-MHD (red) models, the whistler waves asymptote at the electron cyclotron frequency.

For Hall-MHD (blue), it grows without bound and this warrants the need for artificial resistivity. The resistivity sets the cut-off value for the wave-number and all waves higher than this value are not resolved.

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**BrioWu Shock Problem using Ideal MHD and the Two-Fluid Plasma Model**

Left: An MHD shock is initialized with a jump in number density, pressure and magnetic field (note: solutions are not plotted at same times)

Right: The two-fluid solution compared to the ideal MHD solution. Want to see how Hall-MHD and the other reduced fluid models compare to these results.

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Finite electron mass Hall-MHD initial condition and solution for number density Vs x as solution evolves. Certain similarities can be seen between this solution and the ideal MHD and two-fluid solutions.

Once the treatment of the electromagnetic terms is refined for this model, more accurate solutions will be produced.

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Weak Electro-Magnetic Shock Problem using Cold Electron Hall-MHD

Left, Right: A weak shock is initialized in $B_y$ only. Constant density, pressure and $B_x$ used. The solution is evolved for different values of artificial resistivity. The changing wavelength of the waves that are captured is characteristic of whistler waves. As the value of the chosen resistivity decreases significantly, the solution goes unstable.

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Numerical Algorithms Used

- System to be solved:
\[ \frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = S, \]
where \( Q \) = Conserved variables, \( F \) = Flux, \( S \) = Source terms

- Finite Volume Method: High Resolution Wave Propagation method

- Finite Element Method: Runge-Kutta Discontinuous Galerkin method

- Both methods involve solving the Riemann problem across each cell interface

- For the source terms in Hall-MHD and the electro-magnetic terms in the Finite Electron Mass Hall-MHD model, finite differencing is used.

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Conclusions

- The MHD shock problem is presented using ideal MHD, the two-fluid plasma model, Hall-MHD with cold electrons and the finite electron mass Hall-MHD models.

- The dispersion relations are studied to compare the whistler wave for each of these models.

- Hall-MHD is studied and implemented with resistivity. In Hall-MHD the whistler wave grows without bound causing the time step to get smaller and smaller as the solution progresses. Therefore, resistivity is used to cut-off the wave number at a certain value and any waves with wave numbers above this value are not resolved.

- The finite electron mass Hall-MHD model does not have an unbounded wave in its dispersion relation. Also, setting the speed of light to infinity allows for the capture of the longer wavelength, lower frequency waves.
Further Study

- The source terms from Hall-MHD, and the electro-magnetic terms from the finite electron mass Hall-MHD model need to be better treated by projecting them on to basis functions for the discontinuous Galerkin method instead of using finite differencing.

- A better treatment of the electro-magnetic terms as stated above for the finite electron mass Hall-MHD model could potentially lead to solutions that capture two-fluid effects (i.e. have the advantages of Hall-MHD), and resolve all the waves in the system without having to use artificial resistivity to set a cut-off wave number.

- The Magnetic Reconnection problem can then be compared between the models.
Reprints of poster available at: http://www.aa.washington.edu/cfdlab
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