A multi-species 13-moment model for moderately collisional plasmas

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Fluid-based models of collisional transport in multi-species plasmas have typically been applied to parameter regimes where a local thermal equilibrium is assumed. While this parameter regime is valid for low temperature and/or high density applications, it begins to fail as plasmas enter the collisionless regime and kinetic effects dominate the physics. A plasma model is presented that lays the foundation for extending the validity of the collisional fluid regime using an anisotropic 13-moment fluid model derived from the Pearson type-IV probability distribution. The model explicitly evolves the pressure tensor and heat flux vector along with the density and flow velocity to capture dynamics usually restricted to kinetic models. Each particle species is modeled individually and collectively coupled through electromagnetic and collisional interactions. Published by AIP Publishing.

I. INTRODUCTION

Multi-species plasma models are designed to capture dynamics in systems with multiple ion species or charge separation. Each species is treated separately and coupled together through electromagnetic and collisional processes. The focus of this study is to develop a computationally tractable model capable of capturing multi-species behavior with variable thermal equilibration rates.

Thermal equilibration, or thermalization, acts to relax a species toward a local thermal equilibrium by driving its phase space distribution (PSD) to a Maxwellian. The level of thermal equilibrium, or proximity to being Maxwellian, is roughly approximated by the thermalization rate. The Knudsen number Kn = λ_{mp}/L, with mean free path λ_{mp} and characteristic system size L, is one measure of the thermalization rate. For highly collisional systems (Kn ≤ 1), the PSD is considered to be Maxwellian, but as the thermalization rate is reduced to a weaker collisional regime, the PSD can deviate from isotropy resulting in anisotropic transport such as viscosity and thermal conduction. As these anisotropies increase, the validity of the 5-moment model decreases. In the case of plasmas, an imposed magnetic field interacts with collisional transport to drive additional anisotropic behavior collectively known as magneto-viscous effects.

Weakly collisional plasmas, where Kn > 10, are most accurately described by kinetic models. Kinetic models, such as the Boltzmann–Maxwell or particle-in-cell (PIC) models, resolve plasma behavior by evolving the PSD directly using either a six dimensional representation or by sampling the distribution velocity content using a large set of pseudoparticles. These kinetic representations become prohibitively large for capturing macro-scale behavior. Strongly collisional plasmas, where Kn < 10^{-2}, can be accurately described by 5-moment models, which increase the tractability of the problem by reducing the problem’s dimensionality from six to three.

Between the strong and weak collisional regimes exists a collisional transition regime (10^{-2} < Kn < 10) that is not collisional enough for a 5-moment description and is not efficiently modeled by kinetic representations. In this regime, the anisotropic behavior of plasmas is largely influenced by magnetization, wave–particle interactions, advective flow, and intra/interspecies interactions. Numerous moment models have been developed for neutral fluids in the collisional transition regime; however, the development of plasma models for this regime is limited.

Early approaches to higher-moment descriptions of plasmas, such as the 16-moment hydrodynamic formulation in Oraevskii et al., were designed around strongly magnetized, collisionless transport. More recent developments in Wang et al. and Ng et al. have adapted the collisionless Landau closures of Snyder et al. and Goswami et al. to work with a 10-moment plasma model using an approximate form for the heat flux reminiscent of a Bhatnagar–Gross–Krook (BGK) collisional relaxation operator. Additional moment-model approaches to the strongly magnetized, collisionless regime are discussed in Chust and Belmont.

Higher-moment models have also been developed for modeling collisional transport in plasmas. The early work of Kolodner, Herdan and Liley, and Yang describe the development of 13-moment models for capturing collisional transport in mixed gasses and magnetized plasmas based on the 13-moment closure developed in Grad. This work was later extended in Tippett to capture axisymmetric, two-fluid collisional transport in tokamaks. More recently, a two-fluid 10-moment plasma model was developed and implemented numerically in Hakim which is designed to capture collisional transport using a zero heat flux closure derived from the maximum entropy principle of Levermore. This work was extended in Johnson to include a physically accurate description for the heat flux. A one-dimensional, electrostatic form of the two-fluid 13-moment plasma model has also been discussed and implemented in Gilliam; however, it neglects the collisional and magnetization operators required to accurately describe collisional transport in plasmas.

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The model presented here serves as a foundation for high-fidelity, fluid-based plasma models capable of capturing multi-species, magnetized collisional transport near the collisional transition regime without requiring the excessive computational resources of a kinetic description. The multi-species 13-moment model is under development as a computationally inexpensive extension of existing 5-moment\textsuperscript{1-4} and 10-moment models,\textsuperscript{21,27} which incorporates anisotropic plasma transport by evolving both the full pressure tensor and heat flux vector alongside the mass, momentum, and isotropic energy. In this sense, it includes dynamics relating to local thermodynamic equilibrium as well as anisotropic behavior associated with magnetization and collisional transport.

We present the justification and implementation of the 13-moment multi-species model for moderately collisional plasmas in Secs. II–VII. The normalization of the kinetic Boltzmann–BGK–Maxwell model is presented in Sec. II, which is useful for highlighting the importance of the various operators and the resultant plasma behavior. Section III derives and discusses the normalized single-species 13-moment model and its treatment of collisional transport. To illustrate the behavior of the single-species model, it is benchmarked against a shock tube test in Sec. IV. The single-species model is then extended to a multi-species plasma model by including additional effects due to electromagnetic and collisional interactions in Sec. V. A benchmark of the full multi-species 13-moment plasma model against the Hartmann flow problem is presented in Sec. VI. Section VII contrasts the 13-moment plasma model with the more common 5-moment model and highlights the advantages of including additional moments.

II. NORMALIZED BOLTZMANN–MAXWELL MODEL

In kinetic theory, each particle species \( x \) is described by the Boltzmann equation, which, in index notation, is

\[
\frac{\partial f_x}{\partial t} = -v_i \frac{\partial f_x}{\partial x_i} - \frac{q_x}{m_x} E_i \frac{\partial f_x}{\partial v_i} - \frac{q_x}{m_x} \epsilon_{ijk} v_j B_k \frac{\partial f_x}{\partial v_i} + C_x,
\]

with physical space coordinate \( x_i \), velocity coordinate \( v_i \), and Levi–Civita symbol \( \epsilon_{ijk} \). Ampere’s law describes the evolution of the electric field

\[
\frac{\partial}{\partial t} E_i = c^2 \epsilon_{ijk} \frac{\partial B_k}{\partial x_j} - \frac{1}{\epsilon_0} \sum_z q_z \langle v_i f_z \rangle,
\]

where \( c \) is the speed of light, \( \epsilon_0 \) is the permittivity of free space, and the bracket notation

\[
\langle f \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, dv_x \, dv_y \, dv_z,
\]

indicates integration over all of velocity space. The magnetic field evolution is described by Faraday’s law

\[
\frac{\partial}{\partial t} B_i = -\epsilon_{ijk} \frac{\partial}{\partial x_j} E_k.
\]

Finally, the fields are constrained by Gauss’s laws for the electric field

\[
\frac{\partial}{\partial x_j} E_i = \frac{1}{\epsilon_0} \sum_z q_z \langle j_z \rangle,
\]

and the magnetic field

\[
\frac{\partial}{\partial x_j} B_i = 0.
\]

The normalization for this model is based on proton characteristics. The time, length, and velocity scales for the system are normalized by the system’s characteristic time scale \( \tau = \bar{t}/\bar{L} \), length scale \( \bar{L} \), and velocity \( \bar{v} = v/\bar{v} \). For this system, the dimensionless variables are marked with a bar unless otherwise stated. The remaining terms are normalized by reference values

\[
q_z = Z_z e, \quad B_i = B_0 \bar{B}_i, \quad f_z = n_0 \bar{f}_z, \quad m_z = A_z m_p, \quad E_i = E_0 \bar{E}_i,
\]

where \( e \) is the elementary charge, \( m_p \) is the proton mass, and \( n_0 \) is the reference proton number density. The collisional term \( C_z = \nu_p m_p C_z \) is normalized by the reference proton collision frequency

\[
\nu_p = \frac{e^4 n_0 \ln(A)}{3\sqrt{2\pi} a_0^{3/2} \epsilon_0 m_p^{1/2} \bar{v}_p^{3/2} T_0},
\]

with Coulomb logarithm \( \ln(A) \) and reference temperature \( T_0 = m_p \bar{v}_0^2 \). For this study, the Coulomb logarithm is assumed to be species independent and slowly varying in space and time. Applying this normalization to the Boltzmann–Maxwell model yields

\[
\frac{\partial}{\partial t} \bar{f}_z = -\frac{v_0 \tau}{L} \frac{\partial}{\partial x_i} (\bar{v}_i \bar{f}_z) - \frac{eE_0 \tau}{mp v_0 A_z} \bar{E}_i \frac{\partial}{\partial \bar{v}_i} \bar{f}_z - \frac{eB_0 \tau}{m_p A_z} \epsilon_{ijk} \bar{B}_k \frac{\partial}{\partial \bar{v}_i} \bar{f}_z + (v_0 \tau) B_z C_z,
\]

\[
\frac{\partial}{\partial t} \bar{E}_i = \frac{c^2 \bar{B}_0 \tau}{E_0 L} \epsilon_{ijk} \frac{\partial}{\partial x_j} B_k - \frac{e\nu_0 a_0 \tau}{E_0 \epsilon_0} \sum_z Z_z \langle \bar{v}_i \bar{f}_z \rangle,
\]

\[
\frac{\partial}{\partial t} \bar{B}_i = -\frac{eE_0 \tau}{B_0 L} \epsilon_{ijk} \frac{\partial}{\partial x_j} \bar{E}_k,
\]

and

\[
\frac{\partial}{\partial x_j} \bar{E}_i = \frac{\epsilon_0 L}{E_0 a_0} \sum_z Z_z \langle \bar{f}_z \rangle.
\]

To simplify the model, the characteristic time and length scales are related by the characteristic velocity \( v_0 = L/\tau \), while the permittivity of free space and the reference...
magnetic field is related to the proton plasma frequency \( \omega_p = \sqrt{4\pi n_e e^2 / (m_p e^2)} \) and cyclotron frequency \( \omega_c = eB_0 / m_p \). The reference electric field is related to the plasma frequency by \( E_0 = n_0 \omega_p^2 L / e \). Altogether, the normalized Boltzmann equation becomes
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{f}_x = -\frac{\partial}{\partial x_i} \left( \tilde{E}_i \tilde{f}_x \right) - (\omega_c \tau)^2 \frac{Z_x}{A_x} \tilde{E}_i \frac{\partial}{\partial \tilde{v}_1} \tilde{f}_x \\
- (\omega_c \tau)^2 \frac{Z_x}{A_x} \frac{\partial}{\partial \tilde{v}_1} \tilde{f}_x + \left( (\nu_p \tau) \tilde{C}_x \right),
\end{align*}
\] (12)
with Maxwell’s equations
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{E}_i = & - \left( \frac{\omega_c \tau}{\omega_p \tau} \right)^2 \frac{\partial}{\partial x_i} \tilde{E}_k \left( \frac{\partial}{\partial \tilde{v}_1} \tilde{f}_x \right) - \sum_z Z_x \left( \tilde{E}_i \tilde{f}_z \right),
\end{align*}
\] (13)
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{B}_i = & - (\omega_c \tau)^2 \frac{\partial}{\partial x_i} \tilde{B}_k - \sum_z Z_x \left( \tilde{E}_i \tilde{f}_z \right),
\end{align*}
\] (14)
and
\[
\begin{align*}
\frac{\partial}{\partial x_i} \tilde{E}_i = & \sum_z Z_x \left( \tilde{f}_z \right),
\end{align*}
\] (15)
and
\[
\begin{align*}
\frac{\partial}{\partial x_i} \tilde{B}_i = & 0.
\end{align*}
\] (16)
In practice, Eqs. (13) and (14) are modified as presented in Loverich et al.\(^1\) and Hakim et al.\(^2\) to enforce the divergence properties in Eqs. (15) and (16).

The goal behind this normalization is to identify the dominant terms for different plasma regimes. A high plasma density implies a high plasma frequency \( n_0 \sim (\omega_p \tau)^2 \) which undergoes little to no charge separation. A high temperature implies weak collisionality \( T^{3/2} \sim (\omega_p \tau)^2 / (\nu_p \tau) \) which increases the effects associated with local thermodynamic equilibrium, such as viscosity and thermal conductivity. A large cyclotron frequency implies a strongly magnetized plasma \( (\omega_c \tau) \sim B_0 \). The ratio between the various parameters of interest also describes the ordering of the normalization. For instance, the ratio of magnetization to collisionality \( \omega_c / \nu_p \) is a Hall parameter, which determines the importance of magnetically driven anisotropies in a plasma’s viscosity and thermal conductivity.\(^9,30,31\)

While the characteristic velocity in this normalization is directly related to the reference thermal velocity \( v_0 = \sqrt{T_0 / m_p} \), the temporal and spatial scales \( \tau \) and \( L \) are purposefully left undefined. In practice, these scales can be set to normalize the plasma around a specific behavior of interest. For example, setting the temporal scale based on the cyclotron frequency \( \tau = 1 / \omega_c \) rebuilds the other parameters in terms of the magnetization. The plasma frequency parameter becomes \( \omega_p \tau = \omega_p / \omega_c = r_k / \lambda_d \), which specifies the ratio of reference Larmor radius to Debye length, and the collision frequency scale \( \nu_p \tau = \nu_p / \omega_c \) is now a Hall parameter. These magnetically reconfigured normalization parameters, however, have no meaning in an unmagnetized plasma, which is allowed by the underlying model. In summary, this normalization is designed to traverse and order the three main plasma behaviors, namely, charge separation, collisionality, and magnetization, using the plasma parameters \( \omega_p \tau, \nu_p \tau \), and \( \omega_c \tau \). The superbar denoting normalized values is dropped for the remainder of the document.

### III. 13-MOMENT SINGLE-SPECIES MODEL

The 13-moment model describes the evolution of a particle species using 13 scalar moments. For this study, the model is written in balance law form using the first 13 conserved moments of the PSD \( \mathcal{F} \) including the mass density \( \rho = A(\mathcal{F}) \), momentum density \( \rho_i = A(\mathcal{F}) \), energy density tensor \( e_{ij} = A(\mathcal{F}) \), and energy flux vector \( H_i = A(\mathcal{F}) \). The mass density and momentum density can alternatively be described by the number density \( n = \rho / A \) and flow velocity \( u_i = (\rho / A) \). The flow velocity is used to form a random velocity \( \mathcal{W}_i = v_i - u_i \), which defines the fluid’s non-conserved, or primitive, moments such as the pressure tensor
\[
P_{ij} = A(\mathcal{W}_i \mathcal{W}_j) = e_{ij} - \rho \mathcal{W}_i \mathcal{W}_j,
\] (17)
with isotropic pressure \( P = P / A \), and heat flux vector
\[
q_i = \frac{1}{2} A(\mathcal{W}_i \mathcal{W}_j^2) = (H_i - \rho \mathcal{W}_i \mathcal{W}_i - 2u_i P_{ij} - 3u_i P) / 2.
\] (18)
The fluid’s internal energy tensor, or temperature tensor, is defined \( T_{ij} = P_{ij} / n \) and has the isotropic component \( T = T_{ii} / 3 \).

For the 13-moment single-species model, only the intra-species, neutral charge effects of Eq. (12) are examined
\[
\begin{align*}
\frac{\partial}{\partial t} f = & - \frac{\partial}{\partial x_i} \left( v_i f + (\nu_p \tau) C \right).
\end{align*}
\] (19)
The evolution equations for the 13 scalar moments of interest are found by taking velocity moments of Eq. (19). The zeroth moment expresses the conservation of mass
\[
\begin{align*}
\frac{\partial}{\partial t} \rho = & - \frac{\partial}{\partial x_i} \left( \rho \mathcal{W} \right) + (\nu_p \tau) S,
\end{align*}
\] (20)
where \( S = A(\mathcal{C}) \) is the total mass change due to collisions. The first tensor moment describes the conservation of momentum
\[
\begin{align*}
\frac{\partial}{\partial t} \rho_i = & - \frac{\partial}{\partial x_j} \left( \rho \mathcal{W}_j \mathcal{W}_i \right) + (\nu_p \tau) \Phi_i,
\end{align*}
\] (21)
where \( \Phi_i = A(\mathcal{W}_i \mathcal{C}) \) is the total momentum change due to collisions. The second tensor moment describes the energy tensor evolution
\[
\begin{align*}
\frac{\partial}{\partial t} e_{ij} = & - \frac{\partial}{\partial x_k} \left( H_{ijk} + (\nu_p \tau) \mathcal{W}_{ij} \right),
\end{align*}
\] (22)
where \( \mathcal{W}_{ij} = A(\mathcal{W}_i \mathcal{W}_j \mathcal{C}) \) is the total energy change due to collisions. Finally, the evolution of the energy flux vector is
\[
\begin{align*}
\frac{\partial}{\partial t} H_i = & - \frac{\partial}{\partial x_j} \left( G_{ij} + (\nu_p \tau) \mathcal{W}_i \right).
\end{align*}
\] (23)
where $\Theta_i = A(v_i)^2C$ is the change in energy flux due to collisions. For the single-species model, the collision terms $C$ include a collisional relaxation operator to enforce physical thermalization and a diffusive stabilization operator to keep the model stable in the presence of strong anisotropic dynamics. These operators are superimposed to give a collective collisional effect.

As a side note, it is useful to relate the energy equation in Eq. (22) to the 5-moment formulation$^{1-4}$ where the isotropic energy density is defined $e = e_{ij}/2 = (3P + \rho u^2)/2$. The isotropic energy evolution is described by

$$\frac{\partial}{\partial t} e = -\frac{1}{2} \frac{\partial}{\partial v_i} H_{ij} + \frac{3}{2} \langle v_j v_i \rangle \Psi,$$

(24)

where $\Psi = \Psi_{ij}/3$ is the isotropic energy change due to collisions. Since Eq. (24) is implicitly solved when solving Eq. (22), the 13-moment model, along with most other higher-moment models, can be interpreted as an extension of the 5-moment model.

As is seen in Eqs. (20)–(23), each moment of Eq. (19) must evaluate the divergence of a higher-order tensor moment. In Eq. (22), the energy density tensor evolution depends on the energy flux tensor

$$H_{ijk} = A(v_{ij}v_{jk}) = \rho u_i u_j u_k + u_i P_{jk} + u_j P_{ik} + u_k P_{ij} + h_{ijk},$$

(25)

where $h_{ijk} = A(w_{ij}w_{jk})$ is the heat flux tensor. Note that the heat flux vector is $q_i = h_{ij}/2$. The evolution of the energy flux tensor in Eq. (23) depends on the reduced fourth conserved tensor moment

$$G_{ij} = A(v_{ij}v^2) = \rho u_i u_j u^2 + u^2 P_{ij} + 3u_i u_j P + 2u_i u_j P_{ik} + 2u_i u_j q + 2u_k h_{ijk} + g_{ij},$$

(26)

where $g_{ij} = A(w_{ij}w^2)$ is the reduced fourth tensor moment. To truncate this model at 13 scalar moments, the highest unknown moments $g_{ij}$ and $h_{ijk}$ must be defined in terms of the known moments $\rho, u_i, P_{ij}$, and $q_i$.

A. Closure scheme

Closing a moment model means to define the complete set of moments describing the evolution of a species. In the case of the 5-moment model operating near thermal equilibrium, closure is accomplished by defining the full pressure tensor in Eq. (21) and the heat flux vector in Eq. (24) in terms of the mass, momentum, and isotropic energy. This introduces the concept of viscosity and thermal conductivity, where the anisotropic portions of the 5-moment pressure tensor are related to gradients in the velocity field, and the heat flux vector is related to gradients in the temperature.$^5$

Closure schemes become increasingly difficult to derive and evaluate for higher-moment models, which can be seen in Eqs. (17), (25), and (26), where each additional tensor moment increases the complexity of the relation to lower moments. Maximum entropy closures discussed in Levermore,$^6$ Groth and McDonald,$^{10}$ and Johnson$^{27}$ attempt to solve the higher-moment closure problem by assuming a form for the PSD designed to enforce stability; however, these methods tend to require complex iterative methods to solve in models using more than ten moments. More recent developments, discussed in McDonald and Torrilhon,$^{13,14}$ have greatly reduced the numerical costs of the 14-moment maximum entropy closures using an interpolative technique; however, it is not clear if this technique can be extended to the 13-moment model.

Grad methods$^{11}$ treat higher-moment closure schemes by describing deviations from thermal equilibrium using Hermite polynomials. Grad’s original technique for closing the 13-moment model had a tendency to lose stability in the presence of transonic and strong anisotropic flows. The works by Struchtrup and Torrilhon,$^{32}$ Torrilhon,$^{12}$ and Cai et al.$^{13}$ have attempted to regularize the classic Grad closure for the 13-moment model, thereby increasing the region of stability. The 13-moment model closure presented herein attempts to balance the advantages of including additional moments of velocity space with the stability of the closure and the increased complexity of the closure evaluation.

For this study, the closure is taken from Torrilhon$^{1}$ which assumes the PSD is a Pearson type-IV distribution as opposed to the Maxwellian or anisotropic Gaussian distributions used in 5-moment and 10-moment models. The Pearson-type IV distribution is unique in that it can take the form of a Maxwellian distributions,$^7$ as well as incorporate non-symmetric effects that are ignored by Gaussian closures. This highlights the physical motivation for the Pearson IV closure as it is capable of accurately representing fluids in local thermodynamic equilibrium, as well as capturing effects associated with leaving thermal equilibrium. The Pearson-IV closure has been shown to be stable for a wide range of dynamics,$^7$ which when coupled to the simplicity of its evaluation make it an ideal base for constructing a plasma model where strong deviations from thermal equilibrium may arise due to the magnetic and collisional processes.

The goal of the Pearson-IV closure is to include information about the width, skew, and kurtosis of the PSD that arise from leaving thermal equilibrium. The closure defines the two unknown tensor moments $h_{ijk}$ and $g_{ij}$, expressed in Eqs. (25) and (26), in terms of a compact set of closure variables $Y, N_i$, and $K$ which relate to the shape of the Pearson type-IV PSD. Here, $Y$ scales with the magnitude of the distribution’s skew, $N_i$ relates to the skew direction, and $K$ describes the distribution’s kurtosis. In terms of these closure variables, the heat flux tensor can be written

$$h_{ijk} = \frac{1}{2} Y (N_j P_{jk} + N_k P_{ij} + N_i P_{jk} - N_j N_k N_k),$$

(27)

which is purely dependent on the fluid’s density, pressure, and the PSD’s skew. The reduced fourth tensor moment

$$g_{ij} = \frac{1}{4} Y^2 \left( N^2 P_{ij} + 2N_k (N_j P_{jk} + N_i P_{kj}) \right) + \frac{3}{4} Y^2 N_j (P - N^2) + K (3P_{ij} + 2P_{jk} P_{kj}),$$

(28)

shows a direct relationship between the fourth tensor moment and the PSD’s skew and kurtosis.
The closure variables $Y$ and $N_i$ can be defined by taking the trace of Eq. (27)

$$q_i = \frac{1}{2} h_{ij} = \frac{Y}{4} \left( (3P - N^2) \delta_{ij} + 2P_{ij} \right) N_i,$$

(29)

where $\delta_{ij}$ is the Kronecker delta. This is a nonlinear relation between the closure variables $Y$ and $N_i$ and the known primitive moments $\rho$, $P_{ij}$, and $q_i$ which means that to retrieve the closure variables, an iterative solver is required. There are two modifications that can be made to Eq. (29) to simplify the process. The first is to use the definition $N_i = \sqrt{P_{ij} n_j}$ from Torrilhon,\(^7\) where $n_i$ is a unit vector pointing along the direction of skew, to yield

$$q_i = \frac{Y}{4} \left( (3P - P_{kij} n_k n_i) \delta_{ij} + 2P_{ij} \right) \sqrt{P_{jm} n_m}.$$

(30)

The second is to rotate the system to a reference frame where the symmetric tensor $P_{ij}$ is diagonal using orthogonal diagonalization

$$P_{ij} = V_{ik} \Lambda_k V_{jk},$$

(31)

where $V_{ij}$ is the orthogonal rotation matrix and $\Lambda_k$ are the principal pressures. Using this diagonalization, the square root of the pressure tensor is

$$\sqrt{P_{ij}} = V_{ik} \sqrt{\Lambda_k} V_{jk}.$$

(32)

Applying these simplifications results in three coupled equations

$$A_i \left( \frac{x_i x_j \Lambda_j}{x_i x_k} \right) x_i = b_i,$$

(33)

where

$$b_i = 4V_{jk} q_j,$$

(34)

$$x_i = Y V_{jk} n_j,$$

(35)

and

$$A_i (\gamma) = (2\Lambda_i + 3P - \gamma) \sqrt{\Lambda_i}.$$

(36)

The solution to these equations is given by the iterative solver

$$x_i^{r+1} = \frac{b_i}{A_i (\gamma^r)},$$

(37)

with iterative guess

$$\gamma^r = x_i x_j \Lambda_j / x_i^r x_j^r.$$

(38)

At least two iterations are required to converge to an adequate solution\(^7\) with an initial guess of $\gamma^0 = P$. Note that additional iterations are computationally cheap compared to diagonalizing $P_{ij}$.

The closure variables are retrieved as

$$Y = |\tilde{x}|,$$

(39)

and

$$N_i = \frac{1}{Y} V_{ij} x_j \sqrt{\Lambda_j}.$$

(40)

This study uses the “singular closure” of Torrilhon\(^7\) which relates the distribution kurtosis to the magnitude of the distribution skew $K = 1/\rho + Y^2/4$. The combination of $Y$, $N_i$, and $K$ is then used in Eqs. (27) and (28) to find $h_{ij}$ and $g_{ij}$. To implement this closure scheme numerically, additional information about the equation set’s characteristics is required.

1. Characteristic speeds

The 13-moment model is presented in conservative form for use in discontinuous finite element methods (DFEM). Enforcing stability in DFEM, especially for nonlinear hyperbolic systems, depends on how fluxes on discontinuous interfaces between elements are evaluated. These interface fluxes, or numerical fluxes, depend strongly on the characteristics of the flux. Since the Pearson-IV closure scheme is nonlinear and relies on an iterative procedure to solve, a closed form characteristic decomposition of the flux Jacobian is difficult. To counter this, an approximate Riemann solution is used based on the Harten–Lax–Van Leer (HLL) numerical flux.\(^7\)

The HLL flux is designed to circumvent the full characteristic decomposition by approximating the full Riemann solution using two waves instead of a possible thirteen. The HLL flux does not require exact wave speeds; however, discontinuous methods may become excessively diffusive and/or numerically stiff when the wave speeds are poorly approximated.

An HLL numerical flux requires the maximum and minimum characteristic speeds $u_{\text{max}}$ and $u_{\text{min}}$ along the surface normal of a given interface between elements. As an example, the following characteristics are defined for flow along the $x$-axis. For this study, the characteristic speeds are approximated from the one dimensional closure solution defined in Torrilhon\(^7\)

$$Y = \frac{4q_i}{N_i (P_{xx} + 3P)},$$

(41)

where

$$N = \left[ \sqrt{P_{xx}} \quad 0 \quad 0 \right]^T.$$

(42)

Deriving the characteristics is beyond the scope of this paper; however, the result is represented by nine wave speeds $\lambda$. The wave speeds are defined in terms of a thermal velocity $\bar{u} = \sqrt{P/\rho}$, a directional thermal velocity $\bar{u}_{ij} = \sqrt{P_{ij}/\rho}$, and a transport velocity

$$\mu_x = \frac{2q_i}{\rho u_x^2},$$

(43)

where

$$u_x = \sqrt{u_x^2 + 3u^2}.$$

(44)
The four trivial wave speeds are \( \lambda = u_s, \) \( \lambda = u_s + \mu_x, \) and \( \lambda = u_s + \mu_x \pm \sqrt{\mu_x^2 + \mu_y^2}. \) Additional characteristic speeds are given by the roots of the quintic equation
\[
\sum_{n=0}^{5} a_n (\lambda - u_n)^n = 0
\]
where
\[
a_0 = -2A^2(A + 1)\mu_x v_x^4, \quad (45)
a_1 = -A(-3A + B(8A + 6))\mu_x v_x^4, \quad (46)
a_2 = -4(-A(3 + 2A) + B(A + 1))\mu_x v_x^2, \quad (47)
a_3 = 2(-2A(A + 1) + B(2A + 5))\mu_x v_x^2, \quad (48)
a_4 = -2(A + 3)\mu_x, \quad (49)
a_5 = 1, \quad (50)
\]
with \( A = (2 + (P_{xx} + P_{zz})/P_{xx})^{-1} \) and \( B = \mu_x^2/\mu_y^2. \) The extrema roots for this quintic equation can be approximated using the Laguerre-Samuelson inequality
\[
|5(\lambda - u_x) + a_4| < 2\sqrt{6a_4^2 - 10a_5}. \quad (51)
\]
This computationally inexpensive inequality evaluation is adequate for approximating \( u_{\text{max}} = \max(\lambda) \) and \( u_{\text{min}} = \min(\lambda) \) in a HLL flux for applications where the PSD remains near thermal equilibrium. Experimentation has shown that the characteristic speeds in subsonic flows are rarely above three times the thermal velocity.

**B. Evaluating collision operators**

Generally, conservation form collision operators for scattering collisions\(^9,29\) and reactivity\(^33\) are presented in primitive form using the random velocity \( w_i \). In this manner, the total momentum exchange becomes
\[
\Phi_i = A(w_i C) = A(\langle w_i + u_i \rangle C) = R_i + u_i S, \quad (52)
\]
which takes into account both resistive drag \( R_i = A(w_i C) \) and the total mass creation/loss \( S = A(C). \) The total energy exchange can be similarly broken up into
\[
\Psi_{ij} = A(w_i v_j C) = Q_{ij} + u_i R_j + u_j R_i + u_i u_j S, \quad (53)
\]
with collisional heating term \( Q_{ij} = A(w_i w_j C). \) The change in energy flux due to collisions becomes
\[
\Theta_{ij} = A(w_i v_j v_i C) = W_i + 3u_i Q + 2u_i Q_{ij} \nonumber
+ 2u_i u_j R_j + u_i u_j^2 R_i + u_i u_j^2 S, \quad (54)
\]
with heat flux term \( W_i = A(w_i w^2 C) \) and reduced heating term \( Q = Q_{ij}/3. \) This alternative representation for collision integrals simplifies the evaluation of collision operators and makes them more comparable to literature.

**C. Collisional relaxation**

The relaxation collision operator relates to a locally defined collision operator in configuration space that, at most, depends on velocity space gradients and diffusion operators as seen with the Fokker–Planck collision operator.\(^29\) For this study, a simplified collisional relaxation form, known as the Bhatnagar–Gross–Krook (BGK) collision operator
\[
C = -\nu (f - \bar{f}), \quad (55)
\]
with equivalent Maxwellian distribution
\[
\bar{f} = \frac{n}{(2\pi)^{3/2}} \exp \left( -\frac{w^2}{2u^2} \right), \quad (56)
\]
used to describe the relaxation of the Pearson type-IV distribution to a Maxwellian distribution, representing thermal equilibrium, at a rate \( \nu. \) The BGK operator is considered accurate in fluids with small gradients in velocity space,\(^5,21\) which is a rough approximation for moderately collisional plasmas. Further development of linearized Fokker–Planck collision operators, such as those used in the Braginskii\(^9\) model, would be necessary for a more accurate description of collisional transport, and is a topic for future research.

The thermalization rate \( \nu \) is approximated by the normalized Coulomb collision frequency for a species \( \nu_{\text{cc}} \)
\[
\nu = \nu_{\text{cc}} = \frac{n_Z^4}{A_d Z_f^{3/2}}, \quad (57)
\]
By definition, the BGK collision operator does not affect mass \( S = 0, \) momentum \( R_i = 0, \) or isotropic energy \( Q = 0. \) The anisotropic portions of the energy density tensor, however, are affected by scattering collisions
\[
Q_{ij} = -\nu \left( A(w_i w_j f) - A(w_i w_j \bar{f}) \right) = -\nu (P_{ij} - P_{ij}) \delta_{ij}, \quad (58)
\]
which acts to drive the anisotropic pressure components to zero. In the asymptotic limit of thermal equilibrium \((\nu_{\text{cc}} \nu_{\text{cc}} \rightarrow 0),\) the pressure tensor becomes isotropic. Scattering collisions also affect the heat flux
\[
W_i = -\nu \left( A(w_i w^2 f) - A(w_i w^2 \bar{f}) \right) = -2\nu q_i, \quad (59)
\]
to drive the heat flux vector to zero, conducive to thermal equilibrium. Note that these operators are locally defined, and do not depend on the spatial variance in the density, velocity, temperature, or heat flux. The BGK collision operator is designed to describe thermal equilibration effects near thermal equilibrium and as the collisionality decreases \((\nu_{\text{cc}} \nu_{\text{cc}} \rightarrow 0),\) the effect of the collision operator decreases. An additional “collisionless” operator is required for regimes with weaker collisionality.

**D. Diffusive stabilization operator**

In this section, an operator is presented that increases the multidimensional stability of the closure scheme, while mimicking thermal equilibration effects for moderately collisional, magnetized systems. The operator is defined in terms of the intraspecies collisional operators \( S, R_i, Q_{ij}, \) and \( W_i \) to make their application to the model more clear. In order for this stabilization operator to be consistent with the 5-moment
model, the operator has no effect on the mass ($S = 0$), momentum ($R_i = 0$), or isotropic energy ($Q_n = 0$).

The stabilization of the pressure tensor is applied using isotropic diffusion, here represented by an anisotropic heating term

$$Q_{ij} = \frac{\partial}{\partial x_k} \left( D \frac{\partial}{\partial x_k} (P_{ij} - P_{ij}) \right), \quad (60)$$

which helps to keep the pressure tensor positive definite. A positive definite pressure tensor is expected in any physically accurate model; however, the Pearson-IV closure does not explicitly enforce this for systems undergoing strong plasma dynamics. Note that this definition satisfies the isotropic heating condition $Q_{ii} = 0$. To further stabilize the model, isotropic diffusion is also applied to the heat flux

$$W_i = \frac{\partial}{\partial x_j} \left( 2D \frac{\partial}{\partial x_j} q_i \right). \quad (61)$$

The stabilization diffusivity

$$D = \frac{P}{(\nu \tau)^2 \rho \nu}, \quad (62)$$

is analogous to a kinematic viscosity which, by design, reduces to zero in the limit of high-collisionality. As the collisionality decreases, the diffusivity increases, damping the sub-shock structure of the underlying 13-moment closure. It is important to note that this operator is not intended to enforce the correct behavior approaching the collisionless limit as the operator drives the fluid to thermal equilibrium. In practice, the diffusivity is clamped at

$$D = \frac{P}{(\nu \tau)^{\max} \left((\nu \tau)^{\mu}, \tilde{\nu}\right)}, \quad (63)$$

where $\tilde{\nu}$ is a tunable “minimum collisionality.” Setting $\tilde{\nu} = 10$ has been found to provide an adequate amount of diffusion near the collisional transition regime. By limiting the collisionality in this operator, the model retains stability for moderately collisional systems, without driving strong, non-physical collisional transport.

IV. SINGLE-SPECIES RESULTS: ISOTHERMAL SHOCK TUBE

Up to this point, a single-species 13-moment model has been presented which includes thermalization effects. The isothermal shock tube is a test problem for analyzing collisional flow structures in neutral fluids and provides a benchmark to examine the behavior of closures and intraspecies collision operators.

In a highly collisional shock tube, particles congregate into a set of three interfaces known as a shock profile. As the collisionality is lowered, particles can travel further before thermalizing and the shock profile becomes diffuse. The test problem for this study is initialized in thermal equilibrium with a uniform temperature of $T = 1$ across a domain of length 1. The domain is split into high and low pressure regions by an interface at $x = 0$ with left ($l$) and right ($r$) initial conditions $p_l = 8 p_r = 1$ and $u_l = u_r = 0$. The collisionality is controlled by the Knudsen number $Kn = (\nu \tau)^{-1}$ and a normalized collision frequency based on Eq. (57)

$$\nu = \frac{n}{T^{3/2}}. \quad (64)$$

This benchmark compares the 13-moment model against the Boltzmann-BGK model given in Eqs. (19) and (55).

For highly collisional systems, the 13-moment model closure is dominated by the BGK collision operator which drives convergence to the 5-moment fluid model. Figure 1 shows an example for a collisionality of $Kn = 10^{-3}$ which is below the collisional transition regime. For this example, the 13-moment solution (red) does not include diffusive stabilization and shows good agreement with the known solution (dashed blue). The $x > 0$ side of the domain has lower collisionality due to the lower density which leads to a small deviation in the 13-moment heat flux.

At weaker collisionalities, the effect of the BGK collision operator is reduced which reveals the artificial waves of the underlying Pearson-IV closure. This effect is shown in the green trace of Fig. 2 for a moderately collisional test case of $Kn = 10^{-2}$ which is on the border of the collisional transition regime. When the 13-moment fluid model includes the diffusive stabilization operator (red), with frequency cutoff $\nu = 10$, it better matches the Boltzmann–BGK model (dashed blue) in the weaker collisionality region. Increasing the cutoff frequency reveals the artificial waves of the closure, while decreasing the cutoff frequency further tends to drive the system toward local thermal equilibrium, which, in this scenario, would generate a non-physical pressure discontinuity near the front of the shock wave. These results express the need of the diffusive stabilization operator in moderately and weakly collisional systems to damp the artificial, non-physical waves of the Pearson-IV closure; however, this effect must be limited to avoid the non-physical, thermal equilibration effects found when applying excessive diffusion.

V. 13-MOMENT MULTI-SPECIES PLASMA MODEL

The derivation of the 13-moment multi-species model follows the same path as the single-species model where each species is modeled by a separate set of moments. The model is derived from the normalized Boltzmann equation

$$\frac{\partial}{\partial t} f_{sz} = -\frac{\partial}{\partial x_i} (u f_{sz}) - (\omega \tau) \frac{Z_s}{A_s} E_i \frac{\partial}{\partial v_i} f_{sz} - \frac{Z_s}{A_s} \epsilon_{ijk} v_j B_k \frac{\partial}{\partial v_i} f_{sz} + (\nu \tau) C_s. \quad (65)$$

The derivation of the 13 moments of interest remains the operators from Sec. III, and now includes the electromagnetic and interspecies operators developed in Secs. V A–V B.
A. Electromagnetic operators

For the Boltzmann model, the interaction between a charged species and the electromagnetic fields is captured using the Lorentz force operator

$$\frac{\partial f_s}{\partial t} = \frac{Z_s}{A_s} (\omega_p \tau_s)^2 E_i \frac{\partial}{\partial v_i} f_s - \frac{Z_s}{A_s} (\omega \tau) \epsilon_{ijk} v_j B_k \frac{\partial}{\partial v_i} f_s. \quad (66)$$

Taking moments of this operator can be simplified by first identifying that for any moment basis $\langle \vec{v} \rangle$, its moment's evolution is described by

$$\frac{\partial}{\partial t} \langle \phi f_s \rangle = \frac{Z_s}{A_s} (\omega_p \tau_s)^2 E_i \langle \phi \frac{\partial}{\partial v_i} f_s \rangle - \frac{Z_s}{A_s} (\omega \tau) B_k \epsilon_{ijk} \langle v_j \phi \frac{\partial}{\partial v_i} f_s \rangle. \quad (67)$$

Applying integration by parts to the electric field term leads to

$$\left\langle \phi \frac{\partial}{\partial v_i} f_s \right\rangle = \left\langle \frac{\partial}{\partial v_i} (\phi f_s) \right\rangle - \left\langle f_s \frac{\partial}{\partial v_i} \phi \right\rangle. \quad (68)$$

By applying the divergence theorem, and assuming that there are no particles traveling with infinite velocities, this expression simplifies to

$$\left\langle \phi \frac{\partial}{\partial v_i} f_s \right\rangle = - \left\langle f_s \frac{\partial}{\partial v_i} \phi \right\rangle. \quad (69)$$

The magnetic field term is similarly treated

$$\epsilon_{ijk} \left\langle v_j \phi \frac{\partial}{\partial v_i} f_s \right\rangle = - \epsilon_{ijk} \left\langle f_s \frac{\partial}{\partial v_i} (v_j \phi) \right\rangle = - \epsilon_{ijk} \left\langle f_s v_j \frac{\partial}{\partial v_i} \phi \right\rangle - \epsilon_{ijk} \left\langle f_s \phi \right\rangle = - \epsilon_{ijk} \left\langle f_s v_j \frac{\partial}{\partial v_i} \phi \right\rangle. \quad (70)$$

Using this formulation, the zeroth moment ($\phi = 1$) of the Lorentz force operator is

$$\frac{\partial}{\partial t} \rho_s = 0. \quad (71)$$

The first tensor moment ($\phi = v_i$) defines the force density acting on the fluid due to the electromagnetic fields

$$\frac{\partial}{\partial t} P_{i}^{7} = \frac{Z_s}{A_s} (\omega_p \tau_s)^2 E_i \left\langle f_s \frac{\partial}{\partial v_i} v_i \right\rangle + \frac{Z_s}{A_s} (\omega \tau) \epsilon_{ijk} B_j \left\langle f_s v_k \frac{\partial}{\partial v_i} v_i \right\rangle + \frac{Z_s}{A_s} (\omega \tau \phi) \epsilon_{ijk} P_{k}^{7} B_i. \quad (72)$$
The second tensor moment \( \phi = v_i v_j \) defines the electromagnetic coupling to the energy tensor

\[
\frac{\partial}{\partial t} E_{ij} = \frac{Z_x}{A_x} (\omega_x \tau)^2 E_k \left( f_3 \frac{\partial}{\partial v_k} (v_i v_j) \right) + \frac{Z_x}{A_x} (\omega_x \tau) \epsilon_{ijk} B_m \left( f_3 \frac{\partial}{\partial v_m} (v_i v_j) \right)
\]

\[
= \frac{Z_x}{A_x} (\omega_x \tau)^2 \left( p_i^x E_j + p_j^x E_i \right) + \frac{Z_x}{A_x} (\omega_x \tau) \left( \epsilon_{ijk} e_{jk}^x \right) B_i.
\]

The third tensor moment \( \phi = v_i v_j v_k \) defines the electromagnetic coupling to the energy flux vector

\[
\frac{\partial}{\partial t} H^x_{ij} = -\frac{Z_x}{A_x} (\omega_x \tau)^2 E_j \left( f_3 \frac{\partial}{\partial v_j} (v_i v^2) \right) - \frac{Z_x}{A_x} (\omega_x \tau) \epsilon_{ijk} B_k \left( f_3 \frac{\partial}{\partial v_k} (v_i v^2) \right)
\]

\[
= \frac{Z_x}{A_x} (\omega_x \tau)^2 \left( e_{i}^x E_j + 2e_{j}^x E_i \right) - \frac{Z_x}{A_x} (\omega_x \tau) \epsilon_{ijk} H^x_{jk} B_i.
\]

Note that the heat flux vector and anisotropic portions of the energy tensor couple to the magnetic field, similar to what is seen in Braginskii. These electromagnetic operators define long range, long wavelength coupling between particles through electromagnetic effects. For short wavelength interactions between particles, an interspecies collision operator is used.

### B. Interspecies collisions

Interspecies thermalization operators have been well studied for 5-moment plasma models. Generally, interspecies collisions drive the flow velocities and temperatures of the two species together. The rate of thermalization of species \( \alpha \) due to species \( \beta \) is approximated by the interspecies Coulomb collision frequency, which in normalized form is

\[
\nu_{\alpha\beta} = n_{\beta} Z_{\beta}^2 Z_{\alpha}^2 \sqrt{2 \left( 1 + \frac{A_{\alpha}}{A_{\beta}} \right) \left( 1 + \frac{A_{\beta} T_{\beta}}{A_{\alpha} T_{\alpha}} \right)}.
\]

Since scattering collisions do not convert particles from one species to another, there is no mass exchange

\[
S_\alpha = \sum_\beta S_{\alpha\beta} = 0.
\]

The resistive drag operator
\[ R_i^\beta = \sum_\beta R_i^\beta, \quad (77) \]

attempts to drive the velocities of two species together and is approximated by\(^ {29}\)
\[ R_i^\beta = -\nu_{ij} n_j A_j u_i^\beta, \quad (78) \]

where \( u_i^\beta = u_i^\circ - u_i^e \). The collisional heating
\[ Q_{ij}^\beta = \sum_\beta Q_{ij}^\beta, \quad (79) \]
is approximated with
\[ Q_{ij}^\beta = -2\nu_{ij} n_j A_j \left( T_{ij}^\beta - A_\beta u_i^\beta u_j^\beta \right), \quad (80) \]
where \( T_{ij}^\beta = T_{ij}^\circ - T_{ij}^e \). The temperature difference term acts to drive the internal energy of the species together and the second term includes a mass weighting that tends to drive the lower mass species harder than higher mass species as is seen in Braginskii.\(^9\) The isotropic heating due to collisions
\[ Q_{ij}^\beta = \frac{1}{3} Q_{ii}^\beta = -\frac{2}{3} \nu_{ij} n_j A_j \left( 3T_{ij}^\beta - A_\beta u_i^\beta u_j^\beta \right), \quad (81) \]
where \( T_{ij}^\beta = T_{ii}^\beta / 3 \) is the isotropic temperature difference, agrees with the findings of Braginskii\(^9\) and Hinton\(^ {29}\) for low velocity flows in the limit of large mass ratios.

The collisional heat flux exchange
\[ W_i^\beta = \sum_\beta W_i^\beta, \quad (82) \]

attempts to drive the flow of temperature together between species. It is approximated by
\[ W_i^\beta = -\nu_{ij} n_j A_j \left( \frac{q_i^\beta}{n_i^\circ} - \frac{q_j^\beta}{n_j^\circ} \right) + \nu_{ij} n_j A_j \left( 3T_{ij}^\beta - 2u_i^\beta T_{ij}^\circ - u_j^\beta T_{ij}^\circ + 3(A_2 - A_\beta)u_i^\beta u_j^\beta \right), \quad (83) \]

Note that the operators given in Eqs. (78), (80), and (83) are defined to enforce the conservation of total momentum density (\( \Phi_{ij}^\beta = -\Phi_{ij}^{\beta\circ} \)), energy density (\( \Psi_{ij}^\beta = -\Psi_{ij}^{\beta\circ} \)), and energy flux (\( \Theta_{ij}^\beta = -\Theta_{ij}^{\beta\circ} \)).

C. Complete model

With the addition of interspecies collisions and electromagnetic effects, the 13-moment multi-species plasma model contains the terms required for capturing fully ionized plasma transport. To summarize the model developed in Secs. III and V. Each species is described by a continuity equation
\[ \frac{\partial}{\partial t} n_i + \nabla \cdot (u_i^\circ n_i) = 0, \]
\[ \frac{\partial}{\partial t} u_i^\circ + \nabla \cdot (u_i^\circ u_i^\circ + \rho n_i \mathbf{E} - \nabla P) = -\mathbf{J} \times \mathbf{B}, \]
\[ \frac{\partial}{\partial t} E_i + \nabla \cdot (u_i^\circ E_i + \mathbf{J} \times \mathbf{B}) = \epsilon_0 \nabla \cdot \mathbf{D} + \mathbf{J} \cdot \mathbf{E}, \]
\[ \frac{\partial}{\partial t} B_i + \nabla \times (u_i^\circ \times \mathbf{B}) = \mu_0 \nabla \times \mathbf{J}, \]
\[ \frac{\partial}{\partial t} \mathbf{J} + \nabla \times (\mathbf{J} \times \mathbf{B}) = \epsilon_0 \nabla \mathbf{E}, \]
\[ \frac{\partial}{\partial t} \mathbf{D} + \nabla \times (\mathbf{J} \times \mathbf{B}) = \epsilon_0 \nabla \mathbf{E}. \]

The momentum equation
\[ \frac{\partial}{\partial t} p_i^\circ + \nabla \cdot (u_i^\circ p_i^\circ + \rho n_i \mathbf{E}) = -\mathbf{J} \times \mathbf{B} + \mathbf{F}_{\text{coll}} + \mathbf{F}_{\text{ext}}, \]
\[ \frac{\partial}{\partial t} n_i \mathbf{E} + \nabla \cdot (n_i u_i^\circ \mathbf{E} - \rho n_i \mathbf{E} 
abla P) = -\mathbf{J} \times \mathbf{B} + \mathbf{F}_{\text{ext}}, \]
\[ \frac{\partial}{\partial t} B_i + \nabla \times (u_i^\circ \times B_i) = \mu_0 \nabla \times \mathbf{J}, \]
\[ \frac{\partial}{\partial t} \mathbf{J} + \nabla \times (\mathbf{J} \times \mathbf{B}) = \epsilon_0 \nabla \mathbf{E}, \]
\[ \frac{\partial}{\partial t} \mathbf{D} + \nabla \times (\mathbf{J} \times \mathbf{B}) = \epsilon_0 \nabla \mathbf{E}. \]

VI. MULTI-SPECIES RESULTS: HARTMANN FLOW

Benchmarking magnetized plasma transport in moderately collisional systems is a difficult process, and very few tests exist. The Hartmann flow problem is a classic benchmark\(^ {34}\) for resistive, viscous MHD models near the highly collisional regime. The multi-species Hartmann flow, discussed and derived in Appendix A, is an extension of the single-fluid MHD problem.

Similar to a Couette flow, the Hartmann flow problem, shown in Fig. 3, uses two infinite conducting plates to drive shear flow along the y-direction. The shear, along with an imposed magnetic field along x, drives a current in the z-direction which is balanced by resistive forces. The flow generated along z interacts with \( B_z \), to drive a force along y, suppressing, or flattening, the viscous boundary layer.

A viable parameter space for this simulation is \( (\nu_p \tau) = 100, (\omega_p \tau) = 4, (\omega_k \tau) = 100 \), wall velocity \( V = 10^{-3} \), and a domain of \( L = 1 \). The low wall velocity reduces the viscous and resistive heating, while the relatively low
magnetization \((\omega_p \ll \nu_p)\) reduces the anisotropic magnetization effects that the analytical form ignores. A small \(\omega_p \tau\) allows for a larger time step. The fluid is initialized in thermal equilibrium with \(n_s = T_s = B_s = 1\) and \(v_s^2 = 2Vx\), for \(x \in [-0.5, 0.5]\). All other electric and magnetic fields are initialized to zero. The boundary conditions are no-slip, adiabatic, conducting walls.

The results shown in Fig. 4 are for a two-species plasma with masses \(A_s = \{10^{-2}, 1\}\), charges \(Z_s = \{-1, 1\}\), and speed of light \((c/v_0) = 100\) after a normalized time of \(t = 10\). The 13-moment plasma model is solved without diffusive stabilization and is shown to capture the viscous boundary layer of the Couette flow, as well as the flattening of the internal shear profile due to the magnetic forces. The test case has shown that at moderate to high collisionalities and low magnetization, the two-fluid 13-moment plasma model is consistent with the two-fluid 5-moment plasma model.

VII. 13-MOMENT VS 5-MOMENT MODELS

The purpose of increasing the number of moments in the plasma model is to include additional information about the velocity space distribution. In the case of magnetized plasmas, the magnetization drives effects in all full tensor moments above the zeroth moment as shown in Sec. VA. These effects propagate down from higher moments to lower moments through the divergence operator coupling discussed in Sec. III. In this section, the effect of the magneto-viscous coupling from high moments to low moments is shown analytically by comparing the 13-moment and 5-moment plasma models. While interspecies collisions can be important when increasing the number of moments, they are ignored in this section to simplify the comparison.

The multi-species 5-moment plasma model momentum equation

\[
\frac{\partial}{\partial t} \rho_i^s + \frac{\partial}{\partial x_j} \left( \rho_i^s v_j^s + (\omega_p \tau)^s \frac{Z_s}{A_s} \rho_i^s E_j + (\omega_e \tau)^s \frac{Z_s}{A_s} \epsilon_{ijk} \rho_i^s B_k \right) = 0,
\]

is closed by defining the anisotropic energy tensor \(e_{ij}^s\) using a Chapman–Enskog expansion.\(^5\) The derivation begins with the Boltzmann–BGK equation

\[
\mathcal{L}_s[f_s] + (\nu_p \tau)\nu_s(f_s - \tilde{f}_s) = 0,
\]

with the Boltzmann operator

\[
\mathcal{L}_s[f] = \frac{\partial}{\partial t} f + v_i \frac{\partial}{\partial x_i} f + (\omega_p \tau) \frac{Z_s}{A_s} E_i \frac{\partial}{\partial v_i} f + (\omega_e \tau) \frac{Z_s}{A_s} \epsilon_{ijk} v_i B_k \frac{\partial}{\partial v_j} f.
\]

The species distribution is expanded \(f_s = \tilde{f}_s + \frac{1}{(\nu_p \tau)^s} f_{s1}\), where \(f_{s1}\) is a non-thermalized modification of the
Maxwellian distribution $\tilde{f}_x$. Plugging the expansion into Eq. (90) yields

$$f^1_x = -\mathcal{L}_x[f_x].$$

(92)

Using this definition, the total energy tensor is defined as

$$\epsilon_{ij}^T = A_x \{v_i v_j \tilde{f}_x\} = \rho_x u_i^T u_j^T + P_x \delta_{ij} + \Pi_{ij}^T,$$

(93)

with anisotropic pressure tensor

$$\Pi_{ij}^T = \frac{A_x}{(\nu_x \tau)\nu_x} \{v_i v_j \tilde{f}_x\} - \frac{A_x}{(\nu_x \tau)\nu_x} \{v_i v_j \mathcal{L}_x[f_x]\}.$$ (94)

Evaluating the anisotropic pressure moments results in the collisionless evolution of the total energy tensor

$$\Pi_{ij}^T = -\frac{1}{(\nu_x \tau)\nu_x} \left( \frac{\partial}{\partial t} \epsilon_{ij}^T + \frac{\partial}{\partial x_k} H^x_{ij} \right)$$

$$\left[ - (\nu_x \tau)^2 \frac{Z_x}{A_x} \left( p_{ij}^T E_j + p_{ij}^T E_i \right) + (\omega_x \tau) \frac{Z_x}{A_x} \left( \epsilon_{il} e_{jk}^2 + \epsilon_{jk} e_{il}^2 \right) B_i \right].$$

(95)

By relating the total energy density to the pressure and using the conservation of mass and momentum, the anisotropic pressure tensor is rewritten as

$$\Pi_{ij}^T = -\frac{1}{(\nu_x \tau)\nu_x} \left( \frac{\partial}{\partial t} P_{ij}^T + u_k^T \frac{\partial}{\partial x_k} P_{ij}^T + P_{ik}^T \frac{\partial}{\partial x_k} u_k^T \right)$$

$$+ P_{jk}^T \frac{\partial}{\partial x_k} u_k^T + P_{ik}^T \frac{\partial}{\partial x_k} u_k^T + h_{ijk}^T \right)$$

$$+ (\omega_x \tau) \frac{Z_x}{A_x} \left( \epsilon_{il} P_{jk}^T + \epsilon_{jk} P_{il}^T \right) B_i.$$

(96)

Up to this point, all derivations have been exact with respect to the Boltzmann–BGK model. To complete the Chapman–Enskog expansion, an assumption of high collisionality ($\nu_x \tau \gg 1$) and small Hall parameter ($\nu_x \tau \gg \omega_x \tau$) is applied to Eq. (94)

$$\Pi_{ij}^T = -\frac{1}{(\nu_x \tau)\nu_x} A_x \left( v_i v_j \mathcal{L}_x[f_x] \right)$$

$$- \frac{1}{(\nu_x \tau)\nu_x} A_x \left( v_i v_j \mathcal{L}_x[f_x^1] \right)$$

$$\approx -\frac{1}{(\nu_x \tau)\nu_x} A_x \left( v_i v_j \mathcal{L}_x[f_x^2] \right).$$

(97)

This simplification results in a modified form of Eq. (96)

$$\Pi_{ij}^T \approx -\frac{1}{(\nu_x \tau)\nu_x} A_x \left( \delta_{ij} \left( \frac{\partial}{\partial t} P_x + u_k^T \frac{\partial}{\partial x_k} P_x \right) \right)$$

$$+ P_x \frac{\partial}{\partial x_k} u_k^T + P_x \frac{\partial}{\partial x_k} u_k^T + \delta_{ij} P_x \frac{\partial}{\partial x_k} u_k^T \right).$$

(98)

which defines the expansion terms using an isotropic pressure $P_x \rightarrow P_x \delta_{ij}$ and no heat flux $h_{ijk}^T \rightarrow 0$. By removing these higher order collisionality terms, the model requires the plasma to remain near thermal equilibrium and weakly magnetized. The final step of the Chapman–Enskog expansion is to use the evolution of the isotropic pressure

$$\frac{\partial}{\partial t} P_x + u_k^T \frac{\partial}{\partial x_k} P_x = -\frac{2}{3} \frac{\partial}{\partial x_k} \left( \frac{2}{3} \frac{\partial}{\partial x_k} u_k^T \right),$$

(99)

with Eq. (98) to give the familiar anisotropic pressure tensor

$$\Pi_{ij}^T \approx -\frac{P_x}{(\nu_x \tau)\nu_x} \left( \frac{\partial}{\partial x_k} u_k^T + \frac{\partial}{\partial x_k} u_k^T - \frac{2}{3} \frac{\partial}{\partial x_k} \left( \frac{2}{3} \frac{\partial}{\partial x_k} u_k^T \right) \right).$$

(100)

The approximation used in Eq. (97) is the limiting factor for this 5-moment model. The addition of the Hall, or magneto-viscous, term in Eq. (96), which is ignored in Eq. (100), is one example of why the 13-moment model is of interest for magnetized collisional transport. The 13-moment model, along with most other higher-moment models, explicitly takes a portion of these magnetically coupled, collisional terms into account.

**VIII. CONCLUSION**

The 13-moment multi-species plasma model is being developed for capturing magnetized collisional transport in moderately collisional plasmas. The model captures the dynamics of multiple ion species along with electron dynamics and full Maxwell electromagnetics. The model is presented in conservative form for use in discontinuous finite element methods. The Pearson type-IV distribution based closure increases the stability of the model around anisotropic dynamics and, when coupled with intraspecies collision operators, is shown to be consistent with the kinetic Boltzmann–BGK model for moderate and high collisionalities. As the collisionality is lowered, the model tends to diverge from the Boltzmann–BGK model which is due to the lack of a consistent collision operator and closure for the near-collisionless parameter regime. In the limit of a small Hall parameter, the 13-moment multi-species plasma model is shown to be comparable to the 5-moment multi-species plasma model. Overall, this study has found that the 13-moment plasma model has the potential to accurately capture magnetized collisional transport in moderately collisional plasmas in a computationally tractable manner.

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**APPENDIX A: NON-CONSERVED FORM**

The majority of this paper presents the multi-species 13-moment plasma model in balance law form. Balance law form describes the evolution of a conserved quantity, such as mass, momentum, and energy, based on the
divergence of higher moments. These equations can be rewritten in the non-conserved, or primitive, form which directly evolves the number density, flow velocity, pressure, and heat flux. These forms can be written using the convective derivative

$$\frac{d^2}{dt} g = \frac{\partial}{\partial t} g + u_i^j \frac{\partial}{\partial x_i} g.$$  \hspace{1cm} (A1)

The conservation of mass equation becomes

$$\frac{d^2}{dt} n_z = -n_z \frac{\partial}{\partial x_i} u_i^d,$$ \hspace{1cm} (A2)

which assumes no reactivity in the plasma ($S_z = 0$). The evolution of the flow velocity

$$\frac{d^2}{dt} u_i^d = -\frac{1}{\rho_z} \frac{\partial}{\partial x_j} P_{ij} + \frac{Z_a}{A_z} (\omega_e \tau)^2 E_i + \frac{Z_a}{A_z} (\omega_e \tau) \epsilon_{ijk} u_j^d B_k + \left( \nu_p \tau \right) \frac{1}{P_z} \sum_{\beta} R_i^{\beta}.$$ \hspace{1cm} (A3)

is affected by both the electric and magnetic fields, as well as any resistive effects in the plasma. As a side note, since the electric field effect on the Boltzmann model is purely advective in velocity space, it only affects the flow velocity equation. All higher primitive moments do not directly include an electric field term. In the case of the pressure tensor evolution, the magnetic coupling is directly seen

$$\frac{d^2}{dt} P_{ij} = -P_{ik} \frac{\partial}{\partial x_k} u_i^d - P_{jk} \frac{\partial}{\partial x_k} u_j^d - P_{ik} \frac{\partial}{\partial x_k} u_k^d - \frac{2}{3} Q_{ij} \frac{\partial}{\partial x_k} u_k^d.$$ \hspace{1cm} (A4)

Note that this is consistent with the 5-moment multi-species model which evolves the trace of Eq. (A4)

$$\frac{d^2}{dt} P_s = -\frac{2}{3} P_{ij} \frac{\partial}{\partial x_k} u_k^d - \frac{2}{3} Q_{ij} \frac{\partial}{\partial x_k} u_k^d - P_{ik} \frac{\partial}{\partial x_k} u_k^d + \left( \nu_p \tau \right) \sum_{\beta} Q_{ij}^{\beta}.$$ \hspace{1cm} (A5)

The heat flux equation

$$\frac{d^2}{dt} q_i^d = -q_i^d \frac{\partial}{\partial x_k} u_k^d - q_i^d \frac{\partial}{\partial x_k} u_k^d - \frac{1}{2} \mu_{ik} \frac{\partial}{\partial x_k} u_k^d - \frac{1}{2} g_{ik} \frac{\partial}{\partial x_k} u_k^d$$

$$- \frac{1}{2} \frac{\partial}{\partial x_k} g_{ij} + 3 \frac{P_z}{\rho_z} \frac{\partial}{\partial x_j} P_{ij} + \frac{P_z}{\rho_z} \frac{\partial}{\partial x_j} P_{ij}$$

$$+ \frac{Z_a}{A_z} (\omega_e \tau) \epsilon_{ijk} q_k^d B_k$$

$$+ \sum_{\beta} \left( \frac{1}{2} W_i^{\beta} - \frac{3}{2} P_z R_i^{\beta} + 3 P_z R_i^{\beta} \right),$$ \hspace{1cm} (A6)

contains a direct coupling to both the magnetic field and the anisotropic interactions with the plasma resistivity.

APPENDIX B: MULTI-SPECIES HARTMANN FLOW

The multi-species Hartmann flow is a benchmark for magneto-viscous and interspecies collisional effects. The problem setting, shown in Fig. 3, has a plasma set between two infinitely large plates shearing across one another. The plasma travels with the plates at both boundaries, which when coupled with the plasma viscosity, develops an antisymmetric boundary layer profile between the plates.

For the MHD model, the shear velocity profile control is directly dependent on the Hartmann number. For the presented normalization, the profile is largely controlled by $\omega_e \tau$, but must also take into consideration a complex combination of the masses, charges, and thermalization rates. For this derivation, the plasma is assumed to be incompressible and the walls are no-slip and adiabatic. The multi-species Hartmann flow problem is derived from the 5-moment multi-species plasma model where the continuity equation

$$\frac{\partial}{\partial t} n_z = -\frac{\partial}{\partial x_i} (n_z u_i^d)$$ \hspace{1cm} (B1)

and momentum equation

$$\frac{\partial}{\partial t} (A_z n_z u_i^d) = -\frac{\partial}{\partial x_j} (A_z n_z u_i^d u_j^d + P_z \delta_{ij} + \Pi_j^i)$$

$$+ Z_a (\omega_e \tau) n_z E_i + Z_a (\omega_e \tau) n_z \epsilon_{ijk} u_k^d B_k$$

$$- \sum_{\beta} (\nu_p \tau) \nu_{\beta \beta} A_z n_z \left( u_i^d - u_i^d \right),$$ \hspace{1cm} (B2)

use the closure derived in Sec. VII

$$\Pi_j^i = -\frac{P_z}{(\nu_p \tau) \nu_{\beta \beta}} \left( \frac{\partial}{\partial x_i} u_j^d + \frac{\partial}{\partial x_j} u_i^d - 2 \frac{\delta_{ij}}{3} \frac{\partial}{\partial x_k} u_k^d \right).$$ \hspace{1cm} (B3)

Note that the accuracy of this model is limited to small Hall parameters where $\omega_e \ll \nu_p$. The goal of the derivation is to develop a steady state equilibrium. This simplifies the continuity equation

$$\frac{\partial}{\partial x_i} (n_z u_i^d) = 0,$$ \hspace{1cm} (B4)

and momentum equation

$$A_z n_z u_i^d \frac{\partial}{\partial x_j} u_j^d = -\frac{\partial}{\partial x_j} P_z + Z_a (\omega_e \tau) n_z E_i$$

$$+ \frac{\partial}{\partial x_j} \left( \frac{P_z}{(\nu_p \tau) \nu_{\beta \beta}} \left( \frac{\partial}{\partial x_i} u_j^d + \frac{\partial}{\partial x_j} u_i^d \right) \right)$$

$$- \frac{\partial}{\partial x_j} \left( \frac{2 P_z \delta_{ij}}{3 (\nu_p \tau) \nu_{\beta \beta}} \frac{\partial}{\partial x_k} u_k^d \right)$$

$$+ Z_a (\omega_e \tau) n_z \epsilon_{ijk} u_k^d B_k$$

$$- \sum_{\beta} (\nu_p \tau) \nu_{\beta \beta} A_z n_z \left( u_i^d - u_i^d \right).$$ \hspace{1cm} (B5)
Since the problem is designed for a slab geometry, there are no gradients in the $y$ or $z$ directions. Applying the slab geometry to Eq. (B4), with the condition of no flow normal to the walls, implies that $n_a u_a^y = 0$, which further implies $u_a^y = 0$. Steady state Faraday’s Law in a slab geometry ($\frac{\partial}{\partial t} E_y = \frac{\partial}{\partial z} E_z = 0$) is applied in conjunction with the conducting wall boundary condition to give $E_y = E_z = 0$. Applying these conditions to the momentum equation leads to

$$\frac{\partial^2}{\partial x} u_x^y = -\frac{1}{P_x} \left( \frac{\partial}{\partial x} P_x \right) \left( \frac{\partial}{\partial x} u_x^y \right) - \lambda_x u_x^y + \sum_\beta \gamma_{x\beta} \left( u_x^\beta - u_x^\beta \right),$$

(B6)

for the shear flow profile and

$$\frac{\partial^2}{\partial x} u_x^z = -\frac{1}{P_x} \left( \frac{\partial}{\partial x} P_x \right) \left( \frac{\partial}{\partial x} u_x^z \right) + \lambda_x u_x^y + \sum_\beta \gamma_{x\beta} \left( u_x^\beta - u_x^\beta \right),$$

(B7)

for the orthogonal flow. The parameters are defined

$$\lambda_x = (\nu_x) (\omega_x) \frac{Z_a n_a u_x B_z}{P_x},$$

(B8)

and

$$\gamma_{x\beta} = (\nu_x) \frac{A_{x\beta} n_a u_x^\beta}{P_x}.$$  

(B9)

The pressure gradient

$$\frac{\partial}{\partial x} P_x = Z_a (\omega_x) \gamma n_e E_x + Z_a (\omega_x) n_e (u_x^\beta - u_x^\beta B_z),$$

(B10)

scales with the imposed electric field $E_x$, and the magnetic force from the shear and orthogonal flows. By removing the imposed electric field ($E_x = 0$), and assuming a very low Mach flow, the gradient of the pressure is minimized with respect to the background pressure, and can be ignored. This assumption simplifies the equations to

$$\frac{\partial^2}{\partial x} u_x^y = -\lambda_x u_x^y + \sum_\beta \gamma_{x\beta} \left( u_x^\beta - u_x^\beta \right),$$

(B11)

and

$$\frac{\partial^2}{\partial x} u_x^z = \lambda_x u_x^y + \sum_\beta \gamma_{x\beta} \left( u_x^\beta - u_x^\beta \right).$$

(B12)

The small pressure gradient also implies that the parameters $\lambda_x$ and $\gamma_{x\beta}$ are constant with respect to $x$.

To solve for $u_x^y$ and $u_x^z$, Eqs. (B11) and (B12) are written in the matrix form $\vec{\vec{u}}' = A \cdot \vec{\vec{u}}$ where $\vec{\vec{u}}' = \frac{\partial}{\partial x} \vec{\vec{u}}$. For example, if the multi-species velocity vector is given by

$$\vec{\vec{u}} = \begin{bmatrix} u_a^0 & u_a^0 & u_a^1 & u_a^1 & \cdots \end{bmatrix}^T.$$

then the coupling matrix is given by

$$A = \begin{bmatrix} \sum_{\beta \neq 0} \gamma_{0\beta} & \lambda_0 & -\gamma_{01} & 0 & \cdots \\ -\lambda_0 & \sum_{\beta \neq 0} \gamma_{1\beta} & 0 & -\gamma_{10} & \cdots \\ -\gamma_{10} & 0 & \sum_{\beta \neq 0} \gamma_{1\beta} & \lambda_1 & \cdots \\ 0 & -\gamma_{10} & -\lambda_1 & \sum_{\beta \neq 0} \gamma_{1\beta} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$  

To solve, the coupling matrix undergoes an eigendecomposition $A = R \cdot D \cdot R^{-1}$ such that $\vec{\vec{u}}' = R \cdot \vec{\vec{D}} \cdot \vec{\vec{u}}$. Defining an eigenspace velocity $\vec{\vec{s}}' = R^{-1} \cdot \vec{\vec{u}}$, converting the problem into an orthogonal matrix equation $\vec{\vec{s}}'' = D \cdot \vec{\vec{s}}$ with diagonal eigen-value matrix $D$. Each component of $\vec{\vec{s}}$ must obey the boundary value problem $\vec{\vec{s}}'' = D \cdot \vec{\vec{s}}$, which has a general solution

$$s_j(x) = C^0_j e^{\sqrt{D_j} x} + C^1_j e^{-\sqrt{D_j} x}.$$  

(B13)

Given a wall velocity $V$, the boundary conditions are $u^x_j (x_{L/R}) = \pm V$ and $u^y_j (x_{L/R}) = 0$ at boundary positions $x_{L/R}$. The solution coefficients $C^0_j$ and $C^1_j$ are then found by solving Eq. (B13) using the boundary values $\vec{s} (x_{L/R}) = R^{-1} \cdot \vec{\vec{u}} (x_{L/R})$. The real space velocity solution is retrieved using $\vec{\vec{u}} = R \cdot \vec{\vec{s}}$. In general $\{\vec{\vec{u}}, \vec{\vec{D}}\} \subset \mathbb{C}$ making analytical solutions impractically complicated, and for this study, Wolfram Mathematica® was used to find the velocity profiles numerically.

The solution to the magnetic field is given by integrating steady state Ampere’s laws

$$\frac{\partial}{\partial x} B_y = \frac{(\omega_y) e^2}{(e^0)} \sum_x Z_a n_a u_x^z,$$

(B14)

and

$$\frac{\partial}{\partial x} B_z = -\frac{(\omega_y) e^2}{(e^0)} \sum_x Z_a n_a u_x^y,$$

(B15)

with the condition that the plasma is held in a flux conservor

$$\int B_z \, dx = \int B_z \, dx = 0.$$  

(B16)