Semi-implicit Treatment of the Hall Term in Finite Volume, MHD Computations

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Master of Science

University of Washington

1998

Approved by____

(Chairperson of Supervisory Committee)

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Program Authorized
to Offer Degree____

Date____

University of Washington

Abstract

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A finite volume algorithm is developed for solving the Hall term into the time-dependent, non-ideal magnetohydrodynamic (MHD) equations. A semi-implicit method is used to obtain an unconditionally stable algorithm. The semi-implicit operator and the Hall term are adapted to handle complex geometries in finite volumes. A split method is implemented and numerical test results are presented for small-amplitude Hall-MHD waves.

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ACKNOWLEDGMENTS

I am very grateful with my thesis advisor, Professor Uri Shumlak, for his invaluable assistance, guidance, encouragement and friendship throughout my M.S study. The frequent meetings and discussions helped me to develop a deep understanding of my research and of plasma physics in general. I look forward to collaborating with him on future research.

I would like to thank to all the CFD Laboratory, in particular the tremendous help from Bogdan Udrea, who always was there when needed, provinding a a very useful guidance in programming the code. Our innumerable times discussing and questioning the problems, and cheering points of view in general will leave a lasting impression and friendship.

I would also like to thank the faculty, students and staff of the Department of Aeronautics and Astronautics, and of the Department of Applied Mathematics of the University of Washington for providing a stimulating and friendly environment, and invaluable knowledge.

I would also like to thank my family for their constant support, without their help, this would not be possible.

This research was supported by the Air Force Office of Scientific Research.

Chapter 1

INTRODUCTION

If the temperature of a gas is increased beyond a certain limit, it enters a regime where the thermal energy of its constituent particles is enough to overcome the electrostatic forces which bind electrons to atomic nuclei. In such a case, the gas becomes a mixture of charged and neutral particles, in other words, partially ionized. The interaction of the charged particles with the rest of the gas and with externally applied electromagnetic fields leads to a great variety of new physical properties that separates this gas from the other states of matter. The generic name for this hot, partially ionized gas is plasma. The plasma state of matter is defined for a quasineutral mixture of charged and neutral particles that exhibit both individual and collective behavior. Perhaps, their most notable feature is a great electrical conductivity such that externally applied electric fields are effectively canceled inside of them by internally induced currents.

Plasmas are very common. The plasma state dominates the visible universe, as all stars are made of it. The close proximity of the Sun to the Earth permits to study its structure and dynamics in detail. For example, the solar corona expands into space because the Sun's atmosphere is not in static equilibrium. This plasma is known as solar wind and reaches the most distant regions in space. The planets, their moons, and the comets are immersed in the magnetized solar wind dispelling the old view that space was a vaccum. Much of the known matter in the Universe exists as plasmas, therefore, astronomical observations are of great value to understand how natural plasmas work.

On Earth plasmas occur naturally in lightning, lightning balls and the aurora. The most familiar lightning strokes are produced when a charge separation large enough to cause electrical breakdown of air is developed in between a cloud and the ground. An invisible discharge is started near the base of the cloud releasing free electrons towards the ground. When this current approaches the ground, another current moves up from the ground to meet it. Once they have made contact, a visible lightning stroke, called the return stroke, propagates upward from the ground. The explosive heating and expansion of air along the path produces a shock wave that is heard as thunder.

The lightning balls are a little-understood phenomenon. They are generally spherical, from 1 to more than 100 cm in diameter. They usually last less than 5 seconds and move at a few meters per second decaying silently or with a small explosion. They are believed to be natural magnetic dynamos.

Auroral lights are produced when energetic particles are precipitated into the planetary ionospheres, they represent one of the most dynamic products of the interactions of the solar wind with the Earth's magnetosphere.

Plamas have been created in laboratories, for example, in controlled fusion research very hot plasmas of light elements are confined in very strong magnetic fields. Their practical terrestrial applications are extensive, they range from the microfabrication of electronic components to demonstrations of substantial thermonuclear fusion power from magnetically confined plasmas.

In order to study plasmas, a statistical conservation equation, known as the Boltzmann equation, has to be solved for a mixture of neutral and charged particles interacting with electromagnetic fields and Coulomb forces. Extracting appropriately averaged quantities from this kinetic equations produces fluid-type equations that represent the large scale equilibrium and stability of the plasma. This equations are called the magnetohydrodynamic or MHD equations.

The complete MHD equations approximate the behavior of a plasma as a single fluid. The mass, momentum, magnetic induction field and total energy are balanced at each infinitesimal fluid element. This equations add to a total of eight scalar, fully coupled, nonlinear, partial differential equations with mixed terms of parabolic and hyperbolic nature. Nevertheless, a plasma is a far more complex system than the assumptions of magnetohydrodynamics might suggest. The theory has had limited success in explaining phenomena observed in laboratory and space. Its successes include small amplitude waves, MHD stability, and the transport of charge and energy parallel to the magnetic field lines. Its principal failure is the cross-field transport. Deeper kinetic theory, that entails scattering, has proven that second order terms should be keeped to have an accurate description of the cross-field transport.

Our mathematical and physical understanding of the MHD equations is still narrow and to solve them in complex geometries is necessary to ask for the help of a computer. Diverse numerical techniques have been proposed to solve fluid-like equations. The MHD equations are a superset of the Navier-Stokes equations and the same numerical techiniques are, in principle, generalized to the MHD equations. Nevertheless, there are some terms that impose a very restrictive stability limit on the time step and are expensive to compute. An example of this problem arises when the Hall term is not neglected from Faraday's law in the MHD equations. Implicit techniques do not have that stability constraint but are generally associated with very complicated iterative loops. Recently, a semi-implicit technique [HM89] has been developed as an stable way to compute this term. Very accurate results can be obtained if the right semi-implicit operator is chosen and the computational cost can be reduced substantially for computations including this term.

In this thesis, the semi-implicit technique described by Harned and Mikic [HM89] is applied to three-dimensional, finite volume MHD computations with the Hall term included.

In chapter 2 the MHD equations are derived and the conditions over which the

Hall term is not negligible are found. In chapter 3 the semi-implicit technique is applied to a three-dimensional finite volume scheme to handle the physics of the Hall term. In chapter 4 small amplitude MHD waves are studied when the Hall term is not neglected and the dispersion relations are compared to numerical simulations. Finally, chapter 5 contains the conclusions and future work.

Chapter 2

THE MAGNETOHYDRODYNAMIC EQUATIONS

In general, a plasma is a complex system of particles that respond to coupled electromagnetic forces and thermal collisions in such a way that its motion exhibit both individual and collective characteristics.

There are four levels of description for a system of particles in a plasma. The first level is the study of the individual particle orbits. At this level, individual particles are considered and their motion is followed under the influence of the total Lorentz force $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$, where $\mathbf{E}$ is the external electric field, $\mathbf{v}$ and q are the velocity and the electric charge of the particle, and $\mathbf{B}$ is the external magnetic induction field. Helical trajectories along the magnetic field lines and a wide variety of mechanisms causing particles to drift across the magnetic field are discovered using this approach. Nevertheless, this model assumes the presence of an electric field, that can be seen, cannot simply be imposed externally because the electrostatic fields are shielded from the plasma and their effect is confined to a short range characterized by the Debye length. Also, a single particle approach uses the position and the velocity of each particle as variables while this quantities are not measurable or even knowable in reality and therefore their results cannot be compared to experimental data.

The second level of description is obtained by a conservative-statistical approach. The kinetic theory of a plasma is the most fundamental description of the plasma state. The particle species are defined through random distribution functions. All the macroscopic variables as the density, velocity, temperature, etc., can be obtained averaging over the distribution. The conservation of particles in phase space (three-dimensional physical space plus three-dimensional velocity space) is expressed by the

Boltzmann equation 2.7, and the conservative equations for the macroscopic variables can be obtained taking its average values. It is important to follow how the fluid equations are derived from this theory to understand their advantages and limitations. Nevertheless, kinetic theory is much more than an elegant foundation of the fluid approach. It is of great importance due to the low collisionality of many plasmas where the collective behavior is due to electromagnetic coupling of the particles and therefore is common to have distinctly non-Maxwellian distribution functions for each particle specie in the plasma. It is necessary to use kinetic theory to solve this problem because it is impossible to describe it using fluid theory.

The third level of description for a plasma is obtained through the statistical moments of the Boltzmann equation for each particle species in the plasma. Conservation equations for mass, momentum, magnetic induction and energy are obtained for each species. This is a multi-fluid model in which distinct species are treated as individual but interacting continua. Finally, the multi-fluid model can be combined into a single continuous model with averaged properties known as the MHD model. This can be done defining a fluid velocity as that of the center of mass of the multi-fluid model, and a current density as a local difference in the velocity of ions and electrons. This model is useful when the plasma is dominated by collisions, the length scales are long compared to the Debye length, and the frequency is low compared to the plasma oscillation frequency, therefore, contains the physics of the large scale equilibrium and stability of a plasma.

In this chapter, the foundations of the MHD model are reviewed briefly. A general and a simplified MHD model are presented and special attention is devoted to discuss the importance of the Hall term in the generalized Ohm's law.

2.1 The distribution function

For a system of N particles, each one with position and velocity $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, ..., N)$ and N large enough, the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ can be defined such that $f(\mathbf{x}, \mathbf{v}, t)\ dxdydzdv_xdv_ydv_z$ is the number of particles in the volume element $dxdydz$ at position $\mathbf{x}$ and the element $dv_xdv_ydv_z$ in the velocity space at velocity $\mathbf{v}$ and time t. This function belongs to the $(\mathbf{x}, \mathbf{v})$ space that is a six-dimensional phase space.

By definition, the spatial density of particles is

$$n(\mathbf{x}, t) = \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\ d^3\mathbf{v}, \tag{2.1}$$

and the normalized velocity distribution funtion is

$$f'(\mathbf{x}, \mathbf{v}, t) = \frac{f(\mathbf{x}, \mathbf{v}, t)}{n(\mathbf{x}, t)}. \tag{2.2}$$

2.1.1 Maxwellian velocity distribution

If a system is collisional with collision frequency ν, then after a time longer than the collision time $\frac{1}{\nu}$, equipartition of energy by collisions will always cause the system to move towards a Maxwellian velocity distribution

$$f_{maxw} = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp^{\frac{-v^2}{(v_T)^2}}, \tag{2.3}$$

where $v_T = \sqrt{\left(\frac{2kT}{m}\right)}$ is the thermal velocity characterizing the distribution, k is the Boltzmann constant, T is the temperature, m is the mass of each particle and v is the velocity of each particle.

2.2 Macroscopic variables

Measurable macroscopic variables are obtained from the distribution function as velocity "moments". For example, the velocity of the fluid is obtained as the average value of the particle velocity

$$\mathbf{u} = \frac{1}{n}\int_{-\infty}^{\infty} \mathbf{v} f\ d^3\mathbf{v}, \tag{2.4}$$

where $\mathbf{u}$ is the fluid velocity and $\mathbf{v}$ is the particle velocity.

This is, the density of particles n is the zeroth order moment of f, the velocity is the fist order velocity moment of f, the mean particle energy is obtained from the second order moment of f, and so on.

2.3 The Boltzmann equation

Assuming conservation of particles in the phase space, the rate of change of the number of particles in the volume is equal to the net flux of particles into the volume. This can be expressed mathematically as

$$\frac{\partial}{\partial t}\int f d^3\mathbf{x}\, d^3\mathbf{v} = -\int \nabla_{\mathbf{x}} \cdot (\mathbf{v} f)\ d^3\mathbf{x}\, d^3\mathbf{v} - \int \nabla_{\mathbf{v}} \cdot (\dot{\mathbf{v}} f)\ d^3\mathbf{x}\, d^3\mathbf{v}. \tag{2.5}$$

The volume element can be arbitrarily small, then

$$\frac{\partial}{\partial t} f + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f) + \nabla_{\mathbf{v}} \cdot (\dot{\mathbf{v}} f) = 0, \quad \text{or} \quad \frac{df}{dt} = 0. \tag{2.6}$$

Here, $\mathbf{x}$ and $\mathbf{v}$ are independent variables, and if the force $F = m\dot{\mathbf{v}}$ does not depend on the velocity, then, the conservation of particles reduces to the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f = 0. \tag{2.7}$$

If the particles are accelerated by a Lorentz force, this one depends only on the velocity of the normal plane of direction of the acceleration and it is still consistent with the last assumption to get the equation 2.7. In this case the equation 2.7 takes the particular name of Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0. \tag{2.8}$$

A collision source term has to be added to the right hand side of the Botzmann equation 2.7. This term $\bar{C}$ is due to rapid changes in velocity of particles, that either adds particles to $d^3\mathbf{v}$ or removes them. Such changes may be due to high magnetic or electric field fluctuations, or to particle interactions, i.e. collisions. This term is represented as

$$\bar{C} \equiv \left(\frac{\partial f}{\partial t}\right)_c . \tag{2.9}$$

The collisions transfer momentum and energy between particles, but the cannot alter the totality of these properties at any point in space. These constraints require the collision to satisfy

$$\int_{-\infty}^{\infty} \varphi \bar{C} d^3\mathbf{v} = 0, \qquad \left(\varphi = 1, m\mathbf{v}, mv^2\right). \tag{2.10}$$

2.4 The fluid equations

The macroscopic physical quantities are obtained as moments of the distribution function, therefore, the physical equations relating those variables are obtained as moments of the Boltzmann equation 2.7.

2.4.1 Conservation of particles

Taking the zeroth order moment of the Vlasov equation 2.8 leads to

$$\int \frac{\partial f}{\partial t}\, d^3\mathbf{v} + \int \mathbf{v} \cdot \nabla_{\mathbf{x}} f \; d^3\mathbf{v} + \frac{q}{m} \int (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}}\, d^3\mathbf{v} = \int \left(\frac{\partial f}{\partial t}\right)_c d^3\mathbf{v}. \tag{2.11}$$

The third term in the equation 2.11 can be reduced to a vanishing surface integral using Gauss's theorem in velocity space and taking the limit as the surface goes to infinity. The surface area increases as v^2, but practical distribution functions go to zero much faster, for example the Maxwellian distribution 2.3. Also notice that $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\frac{\partial}{\partial \mathbf{v}}$.

The collision term also vanishes if recombination is not considered because the number of particles of the species considered must remain constant.

Then, the equation 2.11, for an arbitrary volume in velocity space, reduces to

$$\frac{\partial n}{\partial t} + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \tag{2.12}$$

which expresses conservation of particles.

2.4.2 Conservation of momentum

Taking the first order momentum of the Vlasov equation means that

$$m \int \mathbf{v} \frac{\partial f}{\partial t}\, d^3\mathbf{v} + m \int \mathbf{v} \left(\mathbf{v} \cdot \nabla_{\mathbf{x}}\right) f\, d^3\mathbf{v} + q \int \mathbf{v} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}}\, d^3\mathbf{v} = \int m\mathbf{v} \left(\frac{\partial f}{\partial t}\right)_c d^3\mathbf{v}. \tag{2.13}$$

The second term can be rewritten as $\nabla_{\mathbf{x}} \cdot \int f \mathbf{v}\mathbf{v} \quad d^3\mathbf{v}$. The velocity is separated into a mean fluid velocity $\mathbf{u}$ and a random thermal velocity $\mathbf{w}$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$ and $\int \mathbf{w} f\, d^3\mathbf{v} = 0$, represented as $\overline{\mathbf{w}} = 0$. Then

$$\nabla_{\mathbf{x}} \cdot \left(n\overline{\mathbf{v}\mathbf{v}}\right) = \nabla_{\mathbf{x}} \cdot \left(n\mathbf{u}\mathbf{u}\right) + \nabla_{\mathbf{x}} \cdot \left(n\overline{\mathbf{w}\mathbf{w}}\right). \tag{2.14}$$

The pressure tensor is defined as $\mathbf{P} = nm\overline{\mathbf{w}\mathbf{w}}$. Where the diagonal elements represent the normal hydrostatic pressure, and the off-diagonal elements describe the transfer of momentum in directions perpendicular to the fluid motion, this is, the effect of the viscosity in the plasma.

The collision term in the equation 2.13 represents the rate of change of momentum density due to collisions between different species i and j, and it is represented by $\mathbf{P}_{ij}$.

Finally, using the conservation of particles 2.12 , the equation 2.13 for an arbitrary volume in velocity space can be reduced to

$$mn \left[\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \nabla\right) \mathbf{u}\right] = qn \left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) - \nabla \cdot \mathbf{P} + \mathbf{P}_{ij}. \tag{2.15}$$

2.4.3 Conservation of energy

The second order statistical moment of the Vlasov equation, multiplied by $\frac{m}{2}$, expresses the conservation of energy for an arbitrary volume in velocity space. This yields to

$$\frac{\partial \epsilon}{\partial t} + \nabla_{\mathbf{x}} \cdot \left[\epsilon\, \mathbf{u} + \mathbf{P} \cdot \mathbf{u} + \mathbf{q}\right] = \mathbf{Q}_{\Omega} - \psi. \tag{2.16}$$

Where the energy density ϵ is defined as the sum of the internal thermal energy, characterized by the random velocity $\mathbf{w}$, and the single-species fluid kinetic energy, characterized by the mean velocity $\mathbf{u}$. The heat flux is represented as $\mathbf{q}$, the ohmic heating is $\mathbf{Q}_{\Omega}$, and ψ is the radiation term.

2.4.4 The multi-fluid model

In the multi-fluid model, the equations for the conservation of particles, momentum and energy are obtained for each specie in the plasma, as done in the previous sections. A fluid of electrons and a fluid of ions is what it is usually used. This description treats different species as individual but interacting continua.

2.4.5 The single-fluid model

The equations used in a multi-fluid model can be added or subtracted to obtain single-fluid equations that contain averaged properties. The density is defined as $\rho = n_i m_i + n_e m_e$, where the subindex denotes ions "i" or electrons " e", this is approximately nm_i for a quasi-neutral mixture of ions and electrons. The single-fluid velocity $\mathbf{v}$ is defined as that of the center of mass, this is approximately $\mathbf{u}_i$, the ion velocity.

The sum of the continuity equations 2.12 for each species yields to the conservation of mass of a quasi-neutral plasma:

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} (\rho \mathbf{v}) = 0. \tag{2.17}$$

Adding the force balance equations 2.15 for each species in the plasma yields to the single-fluid conservation of momentum expressed as

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} \right] = \mathbf{J} \times \mathbf{B} - \nabla_{\mathbf{x}} \cdot \mathbf{P}, \tag{2.18}$$

where the current density is introduced as the difference in charge density transport $\mathbf{J} \approx ne (\mathbf{u}_i - \mathbf{u}_e)$, the total pressure tensor is the sum of those for each species $\mathbf{P} = \mathbf{P}_i + \mathbf{P}_e$, and the sum of the momentum transfer in between species vanishes.

Multiplying the ion equation 2.15 by m_e, the electron equation by m_i and subtracting both equations yields to

$$\frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t} = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{ne}\mathbf{J} \times \mathbf{B} + \frac{1}{ne}\nabla_{\mathbf{x}} \cdot \mathbf{P}_e - \eta \mathbf{J}. \tag{2.19}$$

In order to find this expression of the Ohm's law, the variation of the current density with time, $\frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t}$, is neglected for frequencies lower than the plasma frequency $w_p = \sqrt{\frac{ne^2}{\epsilon_0 m_e}}$, where ϵ_0 is the electric permittivity of the free space. Also, the the momentum transfer from electrons to ions is approximated by $\mathbf{P}_{ei} = -ne\eta\mathbf{J}$, where η is the plasma resistivity. This expression works for most plasmas, nevertheless, the accurate treatment of this problem is still a difficult challenge in plasma physics.

2.5 The MHD equations

The MHD equations can be collected into four groups:

2.5.1 The Maxwell equations

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{2.20}$$

No sources of magnetic field lines

$$\nabla \cdot \mathbf{B} = 0. \tag{2.21}$$

The current density

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}. \tag{2.22}$$

No sources of current density (quatineutrality)

$$\nabla \cdot \mathbf{J} = 0. \tag{2.23}$$

2.5.2 The conservation equations

The conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{2.24}$$

The conservation of momentum

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + \mathbf{P}) = \mathbf{J} \times \mathbf{B} + \rho \mathbf{F}, \tag{2.25}$$

where $\mathbf{F}$ is an external volumetric force (like gravity).

The conservation of energy

$$\begin{aligned} \tfrac{\partial}{\partial t}\left(\rho\left(u + \tfrac{1}{2}v^2\right)\right) + \nabla \cdot \left(\rho \mathbf{v}\left(u + \tfrac{1}{2}v^2\right) + \mathbf{P} \cdot \mathbf{v} + \mathbf{q}\right) \\ = \mathbf{J} \cdot \mathbf{E} + \rho \mathbf{F} \cdot \mathbf{v} - \psi. \end{aligned} \tag{2.26}$$

where ψ is the radiation term and u is the internal energy density.

2.5.3 The thermodynamic relations

The ideal gas equations

$$P = R\rho T = (n_i + n_e)\, kT, \tag{2.27}$$

$$u = \frac{3}{2}\frac{P}{\rho} = c_V T, \tag{2.28}$$

$$\gamma = \frac{c_P}{c_V}, \tag{2.29}$$

and

$$P = \text{const}\ \rho^{\gamma} \quad \exp\left(\ s/c_V\right), \tag{2.30}$$

where γ is the ratio of specific heats c_V and c_P (at constant volume and pressure respectively).

The first law of thermodynamics

$$T\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) s = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) u + P\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)\frac{1}{\rho}. \tag{2.31}$$

2.5.4 The constitutive equations

Ohm's law

$$\mathbf{J} = \sigma \cdot \mathbf{E}' + \beta \cdot \nabla T, \tag{2.32}$$

where β is a laterally isotropic thermoelectric tensor and σ is a second-order, also laterally isotropic, electrical conductivity tensor.

The heat flow

$$\mathbf{q} = -T \left(\beta + \frac{5k}{e} \sigma \right) \cdot \mathbf{E}' - \kappa \cdot \nabla T, \tag{2.33}$$

where k is the Boltzmann constant, e is the charge of the electron and κ is a laterally isotropic thermal conductivity tensor.

The pressure tensor is

$$\mathbf{P} = P\mathbf{I} + \tau = P\mathbf{I} - 2\mu : \nabla^0 \mathbf{v}, \tag{2.34}$$

where μ is a fourth-order, lareally isotropic, viscosity tensor [Woo87], τ is the stress tensor, $\mathbf{I}$ is the identity tensor and the superindex zero means the *deviator* [Woo87] of the tensor $\nabla \mathbf{v}$, this is, its symmetric part with zero trace.

And the generalized electric field

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{ne} \mathbf{J} \times \mathbf{B} + \frac{1}{ne} \nabla \cdot \mathbf{P}_e. \tag{2.35}$$

This equations can be combined to give a closed sistem of eight scalar, fully coupled, non-linear, partial differential equations. This system is written in conservative, non-dimensional form as:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \mathbf{B}\\ e\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\mathbf{v}-\mathbf{B}\mathbf{B}+\left(P+\frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\mathbf{I}\\ \mathbf{v}\mathbf{B}-\mathbf{B}\mathbf{v}\\ \left(e+P+\frac{\mathbf{B}\cdot\mathbf{B}}{2}\right)\ \mathbf{v}-(\mathbf{B}\cdot\mathbf{v})\,\mathbf{B}\end{bmatrix}=$$

$$\nabla\cdot\begin{bmatrix}0\\ (Re\ Al)^{-1}\,\tau\\ (Rm\ Al)^{-1}\,\eta\cdot\nabla\mathbf{B}+\beta_h\left[(\nabla\times\mathbf{B})\,\mathbf{B}-\mathbf{B}\,(\nabla\times\mathbf{B})-\mathbf{P}_e\right]/\rho\\ (Re\ Al)^{-1}\,\mathbf{v}\cdot\tau-(Rm\ Al)^{-1}\,\eta\cdot\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right]+\frac{m_i}{2}\,(Pe\ Al)^{-1}\,\kappa\cdot\nabla T\end{bmatrix}. \quad (2.36)$$

$\mathbf{I}$ is the identity matrix and η is the laterally isotropic, resistivity tensor.

The non-dimensional numbers are defined as follows:

$$\text{Alfvén Number :} \qquad Al \equiv v_A/v, \quad (2.37)$$

$$\text{Reynolds Number :} \qquad Re \equiv Lv/\nu, \quad (2.38)$$

$$\text{Magnetic Reynolds Number :} \qquad Rm \equiv \mu_o LV/\eta, \quad (2.39)$$

$$\text{Péclet Number :} \qquad Pe \equiv LV/\bar{\kappa}, \quad (2.40)$$

$$\text{Hall parameter :} \qquad \beta_h \equiv \frac{1}{\omega_{ci}\tau_A}. \quad (2.41)$$

The characteristic variables are the length L, the fluid speed v, the Alfvén speed $v_A = B/\sqrt{\mu_0\rho}$, the kinematic viscosity ν, the electrical resistivity η, and the thermal difusivity $\bar{\kappa} = \kappa/\rho c_P$. The quantity $\omega_{ci} = \frac{eB}{m_i}$ is the ion cyclotron frequency and $\tau_A = L/v_A$ is the Alfvén transit time.

2.6 The Hall effect in the MHD equations

The vector $\frac{\mathbf{J}\times\mathbf{B}}{ne}$ in Ohm's law 2.19 is known as the Hall or *gyroscopic* term. A comparisson of $\sigma|\frac{\mathbf{J}\times\mathbf{B}}{ne}|$ and $\omega_{ce}\tau_e|\mathbf{J}|$ shows that this term can be dropped if $\omega_{ce}\tau_e \ll 1$. In general, this term can be dropped in either high density or low current regions, but can be important in regions of sharp gradients and low density, like edges.

Im Ohm's law 2.19 , the sum of $\mathbf{v}\times\mathbf{B} - \frac{\mathbf{J}\times\mathbf{B}}{ne}$, given $\mathbf{v} \approx \mathbf{v}_i + \frac{m_e}{m_i}\mathbf{v}_e$ and $\mathbf{J} = ne\left(\mathbf{v}_i - \mathbf{v}_e\right)$, yields to

$$\mathbf{v}\times\mathbf{B} - \frac{\mathbf{J}\times\mathbf{B}}{ne} = \left(\frac{m_e}{m_i} + 1\right)\mathbf{v}_e\times\mathbf{B} \approx \mathbf{v}_e\times\mathbf{B}. \tag{2.42}$$

This shows that when there is Hall effect, i.e. low density and strong magnetic field, the time in between collisions is large compared to the time necessary for a complete rotation of the electron, the plasma electrons are free to gyrate magnetically and they become the total carriers of the magnetic force. The Hall term is important as long as $|\mathbf{v}_i - \mathbf{v}_e|$ is large, and this is generally the case for most plasma physics applications.

The characteristic feature of a plasma described including the Hall term is its ability to exhibit magnetically induced plasma rotation [Wit87]. The angular momentum has to be taken as the usual canonical one plus a field part which is carried by the magnetic field.

A study of MHD waves in unbounded plasmas [Woo87] shows that the Alfvén and magnetosonic branches of the plasma waves can be decoupled only if the Hall term is negligible. Shock wave stability requires the shock to be evolutionary in order to be stable. A shock is said to be evolutionary if the discontinuity responds in a unique manner to small disturbances. The coupling of the Alfvénic and magnetosonic branches, possible only with the inclusion of the Hall effect, prevents one from identifying the intermediate shock as being non-evolutionary.

In cylindrical geometries, the Hall effect gives rise to an azimuthally symmetric, compressible Alfvén-type wave propagation with discrete and infinite spectrum [Goo89]. This wave do not exist if the plasma is incompressible or if the perturbations travel perpendicular to the magnetic field, this is expressed as $\mathbf{k} \cdot \mathbf{B} = 0$, where $\mathbf{k}$ is the wave vector of the perturbation. The Hall effect also reduces the speed of the slow magnetosonic wave and increases that of the fast wave.

Chapter 3

SEMI-IMPLICIT TREATMENT OF THE HALL EFFECT IN FINITE VOLUME COMPUTATIONS

The solution to the MHD equations including the Hall effect is important in regions of a plasma where the velocity of the ions is much different from the velocity of the electrons. The Hall term and the velocity term in the generalized Ohm's law 2.19 can be merged into a single term that only depends on the velocity of the electrons. The resolution of this terms includes, in principle, the resolution of the average electronic motion, and therefore, small scales and a high computational cost. By the other hand, neglecting the Hall term produces the wrong physics. Experimentally, many plasma physics applications have shown that the modification of the MHD equations to include the Hall effect can be important. In reversed-field pinch experiments (RFP), the Hall term can become comparable to the velocity term in Ohm's law. In field-reversed configurations (FRC), some stability effects are explained by rotation induced by the Hall term. In space propulsion, the electromagnetic Hall thruster takes direct advantage of the Hall rotation and current to ionize and accelerate the plasma.

The resolution of the physics associated to the Hall term for time steps comparable to those normally used in MHD computations is a problem related to multiple scale resolution. Explicit schemes for three dimensional time-dependent computations including the Hall effect are complicated by a very restrictive stability margin in the time step. Implicit schemes do not have this restriction but are difficult to implement because the induction equations are fully coupled and the iterative scheme to solve them can be quite complicated.

In 1987, Harned and Mikic [HM89] proposed a semi-implicit technique that allows to compute accurately the physics of the Hall term for those time steps normally implemented in MHD computations. The right expression for the semi-implicit operator was necessary to assure high accuracy at any wave number. Unfortunatelly, they use a spectral method not able to handle complex geometries.

In this chapter, this semi-implicit technique for the Hall term is applied to a three-dimensional, finite volume scheme, able to handle complex geometries. The semi-implicit operator can be expressed in terms of finite differences in between the centered values of the magnetic induction field in each finite volume. The geometric factors, due to the curvilinear transformation, are computed through the volumes and area vectors of the finite volumes.

3.1 Semi-implicit method

A simple view of the semi-implicit method is derived for the coupled system of linear wave equations. The idea is to obtain a second order expression for the time derivative and use it as a dissipation term with a semi-implicit constant to be determined by stability conditions in order to give an unconditionally stable algorithm.

The linear wave equations

$$\frac{\partial u}{\partial t} = a\frac{\partial v}{\partial x} \tag{3.1}$$

and

$$\frac{\partial v}{\partial t} = a\frac{\partial u}{\partial x} \tag{3.2}$$

can be combined to give

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \tag{3.3}$$

this is the same as to write

$$\frac{\partial^2 u}{\partial t^2} - a_0^2\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial x^2} - a_0^2\frac{\partial^2 u}{\partial x^2}, \tag{3.4}$$

so, the semi-implicit advance is defined as

$$u^{n+1} - a_0^2 \Delta t^2 \left[\frac{\partial^2 u}{\partial x^2}\right]^{n+1} = u^n + \Delta t \left[\frac{\partial u}{\partial t}\right]^{n+\frac{1}{2}} - a_0^2 \Delta t^2 \left[\frac{\partial^2 u}{\partial x^2}\right]^{n+\frac{1}{2}}. \tag{3.5}$$

The original equation is used to compute the first derivative averaged over the time step. This technique is found to be better when implemented as a predictor-corrector [HS86] .

3.2 Semi-implicit operator for the Hall term

The semi-implicit operator for the Hall term is obtained differentiating, with respect to time, the linearized Hall term equation

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\frac{1}{\omega_{ci} \tau_A \rho} \nabla \times \left((\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 \right), \tag{3.6}$$

and using $\nabla \cdot \mathbf{B} = 0$ to obtain the second order equation

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = -\frac{1}{\left(\omega_{ci} \tau_A \rho\right)^2} \left[\mathbf{B}_0 \cdot \nabla\right]^2 \nabla^2 \mathbf{B}_0. \tag{3.7}$$

This operator can be simplified to a laplacian but the accuracy of the method is highly decreased for high wave numbers.

3.3 Split method

Harned and Mikic [HM89] studied different ways to implement the semi-implicit operator for the Hall term and found that the approximate laplacian operator is too dispersive for different wave numbers. The complete operator gives accurate results, even at high wave numbers, when implemented as a predictor-corrector.

The split method consists of a predictor stage where an explicit MHD time step with the Hall term is computed. This predicted solution is used to compute the average value of all the variables and to compute a "non Hall" advance of the equations based on the predicted average values $\mathbf{v}^{n+\frac{1}{2}}$ and $\mathbf{B}^{n+\frac{1}{2}}$. Let $\mathbf{B}^*$ be the magnetic field advanced by a "non-Hall" computation. The average of the predicted magnetic field is used to compute the Hall term over the time step. This term is added to the ideal MHD magnetic field $\mathbf{B}^*$ and the semi-implicit operator is added to each side of the equation as in 3.5. The split method gives the following corrector stage:

$$\begin{aligned} &\mathbf{B}^{n+1} + \Delta t^2 \left(\mathbf{C}_H \cdot \nabla\right)^2 \nabla^2 \mathbf{B}^{n+1} \\ = &\mathbf{B}^* + \Delta t^2 \left(\mathbf{C}_H \cdot \nabla\right)^2 \nabla^2 \mathbf{B}^* - \tfrac{\Delta t}{\omega_{ci}\tau_A \rho} \nabla \times \left(\nabla \times \mathbf{B} \times \mathbf{B}\right)^{n+\frac{1}{2}} . \end{aligned} \tag{3.8}$$

The value of the vector constant $\mathbf{C}_H$ is found to assure stability.

In order to obtain second order accuracy in time is necessary to update the rest of the conservative variables based on the averaged value of the magnetic field.

3.4 Finite volume expression for the average Hall term

The Hall term can be rewritten as

$$\nabla \times \left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right] = -\nabla \cdot \left[(\nabla \times \mathbf{B})\,\mathbf{B} - \mathbf{B}\,(\nabla \times \mathbf{B})\right]. \tag{3.9}$$

The finite volume approximation of the volume integral of 3.9 is

$$\int_\Omega \nabla \times \left[(\nabla \times \mathbf{B}) \times \mathbf{B}\right] \; dV = -\sum_{faces} \mathbf{A}_i \cdot \left[(\nabla \times \mathbf{B})\,\mathbf{B} - \mathbf{B}\,(\nabla \times \mathbf{B})\right]_i^{n+\frac{1}{2}} . \tag{3.10}$$

Ω is the control volume, $\mathbf{A}_i$ is the area vector of the $i-th$ face of the control volume. The average Hall term is computed through the average induction field $\mathbf{B}^{n+\frac{1}{2}}$, and the current density $\nabla \times \mathbf{B}$ is approximated by the finite volume expression

$$\nabla \times \mathbf{B} = \frac{1}{Vol\Omega} \sum_{faces} \begin{bmatrix} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{bmatrix}_{faces} . \tag{3.11}$$

Where A_i is the $i-th$ component of the area vector.

Simple average can be used to interpolate from the center of the elements to their faces.

3.5 A finite volume semi-implicit operator for the Hall term

The semi-implicit operator for the Hall term 3.7 has been implemented in cartesian spectral coordinates by Harned and Mikic [HM89]. They found that the crossed terms in the Hall constant vector product $C_H^i C_H^j$ $(for j \neq i)$ are unstable and should be dropped, this changes the operator to

$$(\mathbf{C}_H \cdot \nabla)^2 \nabla_2 \approx \sum_{i=x,y,z} \left(C_H^i\right)^2 \left(\frac{\partial^4}{\partial x^2 \partial i^2} + \frac{\partial^4}{\partial y^2 \partial i^2} + \frac{\partial^4}{\partial z^2 \partial i^2}\right). \tag{3.12}$$

For a regular mesh, this derivatives are computed as

$$\frac{\partial^4}{\partial x^4} = \frac{1}{\Delta x^4}\left[U_{i-2} - 4U_{i-1} + 6U_i - 4U_{i+1} + U_{i+2}\right] + O(\Delta x^2) \tag{3.13}$$

and

$$\begin{aligned}\frac{\partial^4}{\partial x^2 \partial y^2} =& \frac{1}{\Delta x^2 \Delta y^2}\left[U_{j-1,i-1} - 2U_{j,i-1} + U_{j+1,i-1} - 2U_{j-1,i} + 4U_{i,j}\right] \\ &+ \frac{1}{\Delta x^2 \Delta y^2}\left[-2U_{j+1,i} + U_{j-1,i+1} - 2U_{j,i+1} + U_{j+1,i+1}\right]\end{aligned}. \tag{3.14}$$

The same formulas can be used in complex geometries if a coordinate transformation is used to express the operator in curvilinear coordinates. The semi-implicit operator 3.12 transforms to

$$\begin{aligned}(\mathbf{C}_H \cdot \nabla)^2 \nabla^2 \approx& \sum_i \left(\left(C_H^i\right)^2 \frac{\partial^2}{\partial x_i^2}\left(\sum_j \frac{\partial^2}{\partial x_j^2}\right)\right) \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^j}\left(\sqrt{g}\left(C_H^j\right)^2 g^{jk}\frac{\partial}{\partial \xi^k}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}\left(\sqrt{g} g^{il}\frac{\partial}{\partial \xi^l}\right)\right)\right)\end{aligned}, \tag{3.15}$$

using Einstein's tensor notation. The geometric factors $\sqrt{g}$ and g^{ij} are related to the infinitesimal Jacobian matrix of the transformation and are approximated in finite volumes in terms of the areas and volume of each element in the mesh as follows: $vol\Omega = \sqrt{g}$ and $\sqrt{g}g^{ij} = \frac{\mathbf{A}_i \cdot \mathbf{A}_j}{vol\Omega}$.

If locally orthogonal coordinates are used, the operator simplifies to

$$\frac{1}{vol\Omega}\left[\sum_{j=1}^{3}\left(C_H^j\right)^2 \frac{\partial}{\partial \xi^j}\left[\left(\frac{A_j^2}{vol\Omega}\frac{\partial}{\partial \xi^j}\left(\frac{1}{vol\Omega}\sum_{i=1}^{3}\frac{\partial}{\partial \xi^i}\left(\frac{A_i^2}{vol\Omega}\frac{\partial}{\partial \xi^i}\right)\right)\right)\right]\right]. \tag{3.16}$$

This operator looks almost like the one used for cartesian coordinates and does not contain odd crossed terms that cause instabilities.

The matrix generated by this operator has twenty five points in the numerical "molecule" and a simple SOR iterative method [LeV98] is used to invert it and converge to a solution for $\mathbf{B}^{n+1}$. An SOR method is successfull because the matrix to solve is diagonally dominant as the coefficients of U_{ij} reveal in 3.13 and 3.14.

The SOR constant α is grid size dependent and has to be optimized for different geometries. In general, the Hall vector constant $\mathbf{C}_H$ can be computed locally under the conditions found by Harned and Mikic [HM89] : $\mathbf{C}_H \parallel \mathbf{B}_0$ and $\mathbf{C}_H > \frac{B_0}{2\omega_{ci}\tau_A}$, where B_0 is the local magnetic field. This limit is necessary to assure stable modes with large wave number for any value of the Hall parameter $(\omega_{ci}\tau_A)^{-1}$.

3.6 Energy update

The predicted magnetic field $\mathbf{B}^*$ is computed from an MHD advance without the Hall term contribution. After the Hall term is included in a new induction field $\mathbf{B}^{n+1}$, the total magnetic energy predicted needs to be updated adding

$$\Delta\varepsilon = \frac{\left(B^{n+1}\right)^2}{2\mu_0} - \frac{\left(B^*\right)^2}{2\mu_0}. \tag{3.17}$$

Chapter 4

SMALL AMPLITUDE HALL-MHD WAVES

The propagation of small amplitude waves through a plasma is fundamental for several practical reasons. It offers a simple way to compare theoretical results with experiments and numerical results with theory. Once the theory is confirmed, the waves can be used for diagnostic, to understand MHD shock waves, to heat the plasma or to relate unstable waves to the generation of turbulence inside the plasma.

A quite variety of propagation modes can be found in the literature, dispersion relations of quite high order can be obtained, especially when dissipative effects are included such as viscosity and resistivity.

It is possible to start the study of MHD waves with the Appleton-Hartree maneto-ionic theory [Woo87], started over fifty years ago, but it is more natural to start from the MHD equations 2.36.

In general, a hyperbolic, nonsingular, nonlinear system of partial differential equations, can be linearized around a uniform solution using a perturbation series to first order and neglecting all the higher order terms in the equations. If the perturbation is small and given a priori as a general travelling wave, the evolution of the first order perturbation can be solved. An eigenvalue problem appears as the result of the boundary conditions, a discrete frequency spectrum can be obtained and a dispersion relation $\omega = \omega\,(k)$ can be found for each eigenvalue.

In this chapter, Faraday's equation 2.20 is linearized including the Hall term in order to compare the numerical predictions using the semi-implicit operator 3.5 with linearized analytical results for small amplitude MHD waves. The ideal MHD dispersion relations produce three basic modes, known as the fast and slow magnetosonic waves and the Alfvén wave. The Hall term modifies Ohm's law 2.19 in ideal MHD from that for a perfectly conducting plasma to one that is nondissipative, nondiffusive, as in ideal MHD, but that now contains dispersive effects as a result of the Hall effect.

4.1 Linearized induction equation

The ideal induction equation, including the Hall term, is written as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{ne} \mathbf{J} \times \mathbf{B} \right) = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{ne\mu_0} \nabla \times \mathbf{B} \times \mathbf{B} \right), \tag{4.1}$$

and the ideal, electromagnetic, fluid equation is expressed as

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} \nabla \times \mathbf{B} \times \mathbf{B}, \tag{4.2}$$

This equations are linearized around a constant solution of order one adding a first order perturbation of order $\epsilon \ll 1$. The velocity and the magnetic field are expanded as $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$. The following first order correction equations are obtained

$$\frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\mu_0 \rho_0} \nabla \times \mathbf{B}_1 \times \mathbf{B}_0, \tag{4.3}$$

and

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left[\mathbf{v}_1 \times \mathbf{B}_0 - \frac{1}{en\mu_0} \nabla \times \mathbf{B}_1 \times \mathbf{B}_0 \right]. \tag{4.4}$$

This two equations merge into a single second order equation describing the evolution of the magnetic induction

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = \nabla \times \left[\frac{1}{\mu_0 \rho_0} \left(\nabla \times \mathbf{B}_1 \times \mathbf{B}_0 - \frac{1}{en\mu_0} \nabla \times \frac{\partial \mathbf{B}_1}{\partial t} \times \mathbf{B}_0 \right) \right]. \tag{4.5}$$

Now, suppose that the constant field is in the direction of a constant vector $\mathbf{B}_0 = B_0 \mathbf{b}$ and that the first order correction has a wave structure $\mathbf{B}_1 = \bar{\mathbf{B}}_1 \exp\left(i\left(\mathbf{k} \cdot \mathbf{x} - \omega t\right)\right)$, then the equation 4.5 reduces to three linear partial differential equations in space variables, this vector equation is written in nondimensional form as

$$-\bar{\omega}^2\mathbf{B}_1 = \nabla \times \left[\nabla \times \mathbf{B}_1 \times \mathbf{b} + \frac{i\bar{\omega}}{\omega_{ci}\tau_A}\left(\nabla \times \mathbf{B}_1 \times \mathbf{b}\right)\right], \tag{4.6}$$

where $\bar{\omega} = \omega\tau_A$ is the nondimensional frequency for the magnetic induction field perturbation or Hall-MHD waves. The Hall parameter $\beta_h = 1/\left(\omega_{ci}\tau_A\right)$ is multiplying directly the Hall term in the dispersion equation 4.6 .

If a general wave vector $\mathbf{k}$ is considered, the wave dispersion equation 4.6 can be reduced to an equation for the coefficients of $\mathbf{B}_1$, this can be written as

$$-\bar{\omega}^2\bar{\mathbf{B}}_1 = \begin{array}{l} (\nabla + i\mathbf{k}) \times \left[\left(\left(\nabla \times \bar{\mathbf{B}}_1 + i\mathbf{k} \times \bar{\mathbf{B}}_1\right) \times \mathbf{b}\right) \times \mathbf{b}\right] \\ +i\beta_h\bar{\omega}\left(\nabla + i\mathbf{k}\right) \times \left[\left(\left(\nabla \times \bar{\mathbf{B}}_1 + i\mathbf{k} \times \bar{\mathbf{B}}_1\right) \times \mathbf{b}\right)\right]. \end{array} \tag{4.7}$$

4.2 Small amplitude, planar Hall-MHD waves in a bounded plasma

4.2.1 Analytical results

The equation 4.7 is a fully coupled three-component dispersion relation and shows the complicated oscillatory nature of the Hall term.

A manner to simplify this problem is to consider planar waves. A planar wave has a zero wave number perpendicular to the direction of wave propagation, without loosing generality consider $k_x = 0$. The field is allowed to have variations across this direction, thus giving $\bar{\mathbf{B}}_1 = \bar{\mathbf{B}}_1(x)$. Considering a initial zeroth order field of order unity with $\mathbf{b} = \hat{z}$, the equation 4.7 reduces to the following linear system of partial differential equations for the components of the amplitude vector $\bar{\mathbf{B}}_1$:

$$\left(k_z^2 - \bar{\omega}^2\right) B_x = -ik_z\left(\left(\frac{\partial}{\partial x} + \beta_h\bar{\omega}k_y\right) B_z - \beta_h\bar{\omega}k_z B_y\right), \tag{4.8}$$

$$\left(k_z^2 - \bar{\omega}^2\right) B_y = k_z\left(\left(\beta_h\bar{\omega}\frac{\partial}{\partial x} + k_y\right) B_z - i\beta_h\bar{\omega}k_z B_x\right), \tag{4.9}$$

$$\left(k_y^2 - \frac{\partial^2}{\partial x^2} - \bar{\omega}^2\right) B_z = k_z \left(i \left(\beta_h \bar{\omega} k_y - \frac{\partial}{\partial x}\right) B_x + \left(k_y - \beta_h \bar{\omega} \frac{\partial}{\partial x}\right) B_y\right). \quad (4.10)$$

The elimination of the B_x and B_y components can be simplified if the condition $\nabla \cdot \mathbf{B} = 0$ is imposed. This condition yields to the following equation:

$$\frac{\partial B_x}{\partial x} + ik_y B_y + ik_z B_z = 0. \quad (4.11)$$

After the B_x and B_y components are eliminated, the following harmonic equation is found for B_z,

$$\frac{\partial^2}{\partial x^2} B_z = \left[k_y^2 + \frac{\left(k_z^2 - \bar{\omega}^2\right)^2 - \left(\beta_h \bar{\omega} k_z^2\right)^2}{\left(k_z^2 \left(1 - \left(\beta_h \bar{\omega}\right)^2\right) - \bar{\omega}^2\right)}\right] B_z. \quad (4.12)$$

The other two components are obtained in terms of B_z as

$$B_y(x) = \frac{k_z \left(k_z^2 k_y \left(1 - \left(\beta_h \bar{\omega}\right)^2\right) - \bar{\omega}^2 \left(\beta_h \bar{\omega} \frac{\partial}{\partial x} + k_y\right)\right)}{\left(\left(k_z^2 - \bar{\omega}^2\right)^2 - \left(\beta_h \bar{\omega} k_z^2\right)^2\right)} B_z(x), \quad (4.13)$$

and

$$B_x(x) = \begin{array}{l} \frac{-ik_z}{(k_z^2 - \bar{\omega}^2)} \left[\frac{\partial}{\partial x} + \beta_h \bar{\omega} k_y\right] B_z(x) \\ + \frac{ik_z}{(k_z^2 - \bar{\omega}^2)} \left[\beta_h \bar{\omega} k_z^2 \left(\frac{k_z^2 k_y \left(1 - (\beta_h \bar{\omega})^2\right) - \bar{\omega}^2 \left(\beta_h \bar{\omega} \frac{\partial}{\partial x} + k_y\right)}{\left((k_z^2 - \bar{\omega}^2)^2 - (\beta_h \bar{\omega} k_z^2)^2\right)}\right)\right] B_z(x). \end{array} \quad (4.14)$$

The imaginary number i reflects a ninety degree phase difference. It is also convenient to define the coefficient in the right hand side of the equation 4.12 as

$$\nu^2 = -\frac{\left(k_z^2 - \bar{\omega}^2\right)^2 - \left(\beta_h \bar{\omega} k_z^2\right)^2}{k_z^2 \left(1 - \left(\beta_h \bar{\omega}\right)^2\right) - \bar{\omega}^2}, \quad (4.15)$$

then, the solution of 4.12 is obtained as

$$B_z(x) = A_1 \cos\left[x\sqrt{\nu^2 - k_y^2}\right] + A_2 \sin\left[x\sqrt{\nu^2 - k_y^2}\right]. \quad (4.16)$$

Experimental tests of MHD theory are frequently undertaken using bounded plasmas where conductor plates are used as boundary conditions. The boundary condition for a perfectly conducting wall requires that the magnetic field lines are continuous at the wall. This condition eliminates the the magnetic field in the direction normal to the wall, therefore, $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$. Also $\hat{\mathbf{n}} \times \mathbf{E} = 0$ needs to be imposed at the wall. This condition applied to Ohm's law gives $\hat{\mathbf{n}} \cdot \mathbf{v} = 0$ and $\hat{\mathbf{n}} \cdot \mathbf{J} = 0$ at the conductor walls.

In order to simulate a bounded plasma, conductor boundary conditions are used asking for $B_x = 0$ and $B_y = 0$ at $x = 0.5$ and $x = -0.5$. In general, a single equation for B_x must be found and the boundary conditions must be imposed over the solution. The condition is satisfied in general for a function $f\left(k_y, k_z, \omega\right) = 0$.

For simplicity, the case of decoupled modes is solved setting $k_y = 0$. This condition simplifies the system of equations 4.8, 4.9 and 4.10, to the following system:

$$\left(k_z^2 - \bar{\omega}^2\right) B_x = -ik_z \left[\frac{\partial B_z}{\partial x} - \beta_h \bar{\omega} k_z B_y\right], \tag{4.17}$$

$$\left(k_z^2 - \bar{\omega}^2\right) B_y = k_z \beta_h \bar{\omega} \left[\frac{\partial B_z}{\partial x} - ik_z B_x\right], \tag{4.18}$$

$$\left(\frac{\partial^2}{\partial x^2} + \bar{\omega}^2\right) B_z = k_z \left[i\frac{\partial B_x}{\partial x} + \beta_h \bar{\omega} \frac{\partial B_y}{\partial x}\right]. \tag{4.19}$$

And the equation 4.11 simplifies to $\frac{\partial B_x}{\partial x} = -ik_z B_z$.

The elimination of two of the components yields to the same harmonic equation for B_x and B_z, this is

$$\frac{\partial^2 B_x}{\partial x^2} = -\nu^2 B_x. \tag{4.20}$$

The conductor boundary conditions can be imposed to get the solution

$$B_x\left(x\right) = \epsilon \cos\left(\nu x\right), \tag{4.21}$$

with eigenvalues $\nu = \pi, 2\pi, \cdots$. The solution 4.21 is used to find the expressions for B_z and B_y, these are

$$B_z\left(x\right) = -\frac{i\epsilon\nu}{k_z} \sin\left(\nu x\right), \tag{4.22}$$

and

$$B_y\left(x\right)=-\frac{i\epsilon\beta_h\bar{\omega}\left(\nu^2+k_z^2\right)}{\left(k_z^2-\bar{\omega}^2\right)}\cos\left(\nu x\right). \tag{4.23}$$

Taking the real part of the solution yields to the following expression for decoupled, small amplitude, MHD waves:

$$\mathbf{B}\left(x,y,z,t\right)=\begin{pmatrix}0\\0\\B_0\end{pmatrix}+\epsilon\begin{pmatrix}\cos\left(\nu x\right)\cos\left(k_z z-\bar{\omega}t\right)\\ \frac{\beta_h\bar{\omega}\left(\nu^2+k_z^2\right)}{\left(k_z^2-\bar{\omega}^2\right)}\cos\left(\nu x\right)\sin\left(k_z z-\bar{\omega}t\right)\\ \frac{\nu}{k_z}\sin\left(\nu x\right)\sin\left(k_z z-\omega t\right)\end{pmatrix}. \tag{4.24}$$

This is a travelling wave in the $\hat{z}$ direction.

The dispersion relation is found to have the following, commonly known expression

$$\left[\beta_h\bar{\omega}\right]^2=\left(1-\frac{\bar{\omega}}{k_z^2}\right)\left(1-\frac{\bar{\omega}^2}{k_z^2+\nu^2}\right). \tag{4.25}$$

The equation 4.25 is a second order algebraic expression for $\bar{\omega}^2$ that can be solved as

$$\begin{aligned}\bar{\omega}^2=\tfrac{1}{2}\left[\left(\nu^2+k_z^2\right)\left(\beta_h^2k_z^2+1\right)+k_z^2\right]\\ \pm\left[\tfrac{1}{4}\left(\left(\nu^2+k_z^2\right)\left(\beta_h^2k_z^2+1\right)+k_z^2\right)^2-k_z^2\left(\nu^2+k_z^2\right)\right]^{1/2}.\end{aligned} \tag{4.26}$$

The dispersion relation for the fast or slow magnetosonic waves is obtained choosing the positive or the negative sign in the equation 4.26, respectively.

The dispersion relation for the fast and slow modes with the Hall term is plotted in Figure 4.1 . The value of $\omega_{ci}\tau_A=3.5$ is chosen because it corresponds to the size of the Hall term in ZT-40M reversed-field pinch experiments.

4.2.2 Numerical results

The numerical solution of small amplitude Hall-MHD waves is compared to the analytical result obtained in the previous section. The semi-implicit technique developed in Chapter 3 is used inside the finite volume *warp*3 code [OSJE97] to include the Hall

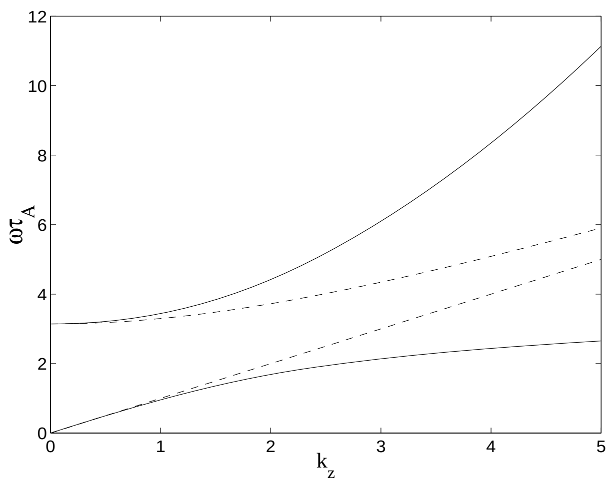

Figure 4.1: Dispersion relation for the fast and slow modes with $\omega_{ci}\tau_A = 3.5$. The dashed lines represent the ideal MHD modes.

terms into the MHD equations. The equations are advanced splitting the ideal MHD part and the Hall term contribution.

A rectangular, planar mesh of 30×30 points in the x and z directions is used. Conductor boundary conditions $(\mathbf{B}\cdot\mathbf{n} = 0, \mathbf{J}\cdot\mathbf{n} = 0)$ are used in the x direction and periodic boundary conditions are used in the z direction. The size of the domain is adjusted in the z direction for each wave number. The value of $\omega_{ci}\tau_A = 3.5$ is chosen and therefore the Hall semi-implicit vector constant is set to $\mathbf{C}_H = 0.15$, in agreement with the stability condition $\mathbf{C}_H > \frac{B_0}{2\omega_{ci}\tau_A}$.

The explicit analytic expression for the Hall term, given the initial magnetic field 4.24 at $t = 0$, is computed as

$$\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] = \begin{bmatrix} \frac{\epsilon k_z^2 \beta_h \bar{\omega}\left(\nu^2 + k_z^2\right)}{(k_z^2 - \bar{\omega}^2)} \cos(\nu x) \sin(k_z z) \\ -\epsilon\left(k_z^2 + \nu^2\right)\cos(\nu x)\cos(k_z z) \\ \frac{\epsilon k_z \nu \beta_h \bar{\omega}\left(\nu^2 + k_z^2\right)}{(k_z^2 - \bar{\omega}^2)} \sin(\nu x)\cos(k_z z) \end{bmatrix}. \qquad (4.27)$$

As a first test, this expression is compared numerically to its finite volume approximation 3.10 at $t = 0$. Second order accuracy in space is obtained as expected.

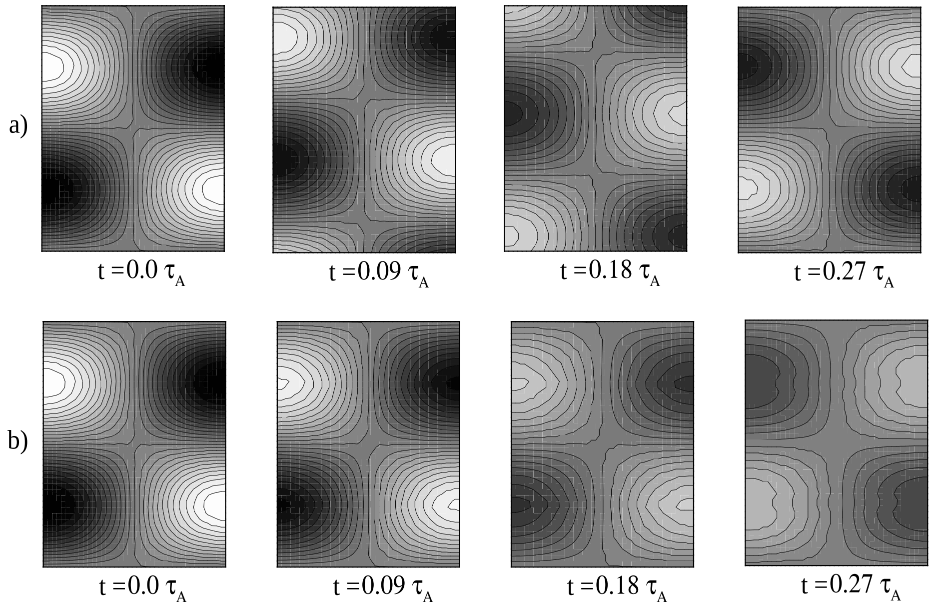

Figure 4.2: Contours of B_z with wave number $k_z = 5.0$. a) Hall-MHD magnetic induced rotation. b) Ideal MHD.

Qualitatively, it is observed that the Hall effect produces magnetic induced rotation as shown in Figure 4.2. This behavior does not appear in ideal MHD.

Less than first order accuracy in time is obtained with a simple splitting technique and the results are quite disappointing even for relatively high resolution runs.

For any value of $\epsilon \ll 1$, a numerically predicted frequency of 6.95 is obtained given $k_z = 5.0$, against 11.1 predicted by the linear theory. Figure 4.3 presents the points given by the numerical scheme. The measured frequency is off independent of grid spacing and time step size.

It is important to know that an explicit run (without the semi-implicit operator) gives the same results. This proves that the semi-implicit operator has little influence over the numerical result once the Hall term is added. The error in the frequency has the same effect of reducing the Hall parameter $(\omega_{ci}\tau_A)^{-1}$ by a half for any value of the Hall parameter.

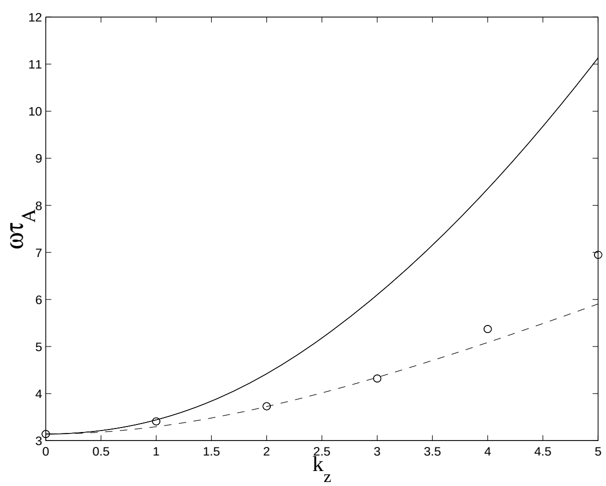

Figure 4.3: Dispersion for the fast mode with $\omega_{ci}\tau_A = 3.5$. The points represent the numerical results. The dashed line represents the dispersion without Hall effect.

Chapter 5

THESIS CONTRIBUTION AND FUTURE RESEARCH

5.1 Thesis contribution

In this thesis, the semi-implicit operator found by Harned and Mikic [HM89] was adapted to finite volume, conservative schemes. The full operator is able to handle complex geometries and its geometric properties are expressed in terms of the area vectors and volume of the finite elements. It is found that the semi-implicit method gives the same results as an explicit advance of the equations. An unconditionally stable method is obtained able to handle any size of time steps.

5.2 Difficulties

The numerical results are quite disappointing even for high resolution runs. Exhaustive work was done checking the accuracy of the finite volume approximation of the Hall term. It is found that the semi-implicit method gives the same result as the explicit method but without the stability conditions over the time step.

The Hall term is of hyperbolic, dispersive nature. In this thesis, this term was handled as a source term. Normally, this approach works for parabolic terms but it may be not accurate for hyperbolic ones. Harned and Mikic [HM89] treated it as a source term in spectral representation and it is not clear if the same scheme should work for finite volumes. Perhaps this is the reason for the error. If this is the case, in order to avoid this problem and still being able to handle any time step, this method can be combined with a Hall-MHD Riemann solver in the predictor step to get the correct wave structure. Then the semi-implicit method can be used as described by

Harned and Schnack [HS86]. The development of a Hall-MHD Riemann solver is still matter of future research.

5.3 Future research

The applications of an MHD solver with the Hall terms are extensive. The method presented in this thesis can be useful for steady state solutions where the transients are not important. As an example, the Hartmann-Hall flow can be studied numerically, theoretical results have been obtained by Waleffe [Wal84], and the velocity profiles can be matched.

The development of accurate finite volume methods able to handle the Hall term are very important to the study of ion acceleration with close electron drift [Kau84], theoretical base for the Hall thruster. The Hall terms are also important for new fusion research projects, where the Hall term are important around edges or where the Hall rotation is a factor of stability. New concepts of non-thermal fusion have been based on the dynamo action produced by the Hall effect [Wit88].

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Appendix A

SUBROUTINE INPUTS AND OUTPUTS

In this section, the new subroutines included in the $warp3$ code, as part of the research to handle the physics of the Hall term, are explained.

One main subroutine $hall - main$ is included in the code. This routine calls the functions $hall - average$, $hall - curlb$, $hall - source1$, $hall - source2$, $hall - source3$ and the subroutines $hall - bc$ and $hall - geometry$. A new module of parameters is introduced with the name $hall - inp$ referring to inputs. An specific initialization routine $init - hall$ was used for the dispersion relation matching.

A.1 Subroutine $hall - main$

$Hall - main$ is the routine that adds the average Hall term to the induction equation and applies the semi-implicit operator to obtain an unconditionally stable algorithm. New parameters are defined in the module $hall - param$: the logical flag $hall - flag$ turns on the routine; the Hall parameter $(\omega_{ci}\tau_A)^{-1}$ receives the name of $hallp$; the semi-implicit vector constant has components $challi$, $challj$, and $challk$ (oriented in the generalized directions of the grid); the SOR constant is $omega$ and the SOR iterations are $hiter$; and, specifically for the Hall-MHD waves, the parameters nu are the eigenvalue, kz is the wave number and $epsil$ is the amplitude of the perturbation. Inside $hall - main$ the computations start with the average of the magnetic field components on every face of every direct neighboring cell. This values are computed in the routine $hall - average$ and stored in the dummy variables $bij - ave$, where i denotes de face and j denotes the cartesian component. The average values are used to compute the current density on every direct neighboring cell, their values are computed by the function $hall - curlb$ and stored in $curlb(i, j)$ where i denotes the face and j denotes the cartesian component. The average of the current density is computed in each face of the working cell in order to be used by the function $hall - source1$ to compute the integral of the Hall term over the finite volume, using Gauss' theorem. The dummy variable $bstari$, where i denotes the cartesian component, is used to store the input magnetic field. The routine $hall - boundary$ is used to extrapolate the values of this variables to the ghost cells along the boundaries.

The average Hall term is added to each component of the magnetic field. The semi-implicit operator applied to the input magnetic field is also computed there in the function $hall - source2$ and added to the magnetic field components to be stored in the dummy variables $shalli$, where i denotes the cartesian component. The geometric factors needed inside the semi-implicit operator are computed in the routine $hall - geometry$ and stored in the dummy variable $geom(i, j)$, where i denotes the neighbor, and j denotes the local face.

The SOR method is used to invert the semi-implicit operator applied to the new magnetic field. Inside the function $hall - source3$ the off-diagonal elements of the operator are used as a source term and are updated after each iteration of the diagonal elements. The iteration index is m and the number of iterations is $hiter$.

It was found that the SOR method converges very rapidly for values of $omega = 1.35$. But this value is grid-dependent and needs to be optimized for different geometries.

The converged values are dumped to the conserved variables q, the boundary conditions are imposed and the magnetic energy is updated.

This routine is able to handle parallel runs.

In summary, the routine receives the values of the ideal MHD update of the magnetic field, computes the average Hall term with them, adds it to the equations and applies the semi-implicit method to return a three new components of the magnetic field with updated boundary conditions and energy.