Implicit Time-Integration Schemes for Finite Element Two-Fluid Plasma Code

B. Srinivasan, U. Shumlak

Aerospace and Energetics Research Program
University of Washington

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Abstract & Motivation

- Two-fluid model: Fluid equations for the ions and electrons, Maxwell’s equations for the electric and magnetic fields
- Two-fluid effects become significant when
  - characteristic spatial scales are on the order of the ion skin depth
  - characteristic time scales are on the order of the inverse ion cyclotron frequency
- Hall and diamagnetic drift terms capture the two-fluid physics
- Explicit time-stepping schemes can be very restrictive due to speed of light and the electron plasma frequency
- Provides motivation to study implicit time-stepping schemes
Collisionless fluid equations are used for ions and electrons denoted by subscript $s = \{e, i\}$.

\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s u_s) = 0
\]

\[
\frac{\partial \rho_s u_s}{\partial t} + \nabla \cdot (\rho_s u_s u_s + \nabla p_s) = \frac{\rho_s q_s}{m_s} (E + u_s \times B)
\]

\[
\frac{\partial \epsilon_s}{\partial t} + \nabla \cdot ((\epsilon_s + p_s) u_s) = \frac{\rho_s q_s}{m_s} u_s \cdot E
\]

\[
\epsilon_s \equiv \frac{p_s}{\gamma - 1} + \frac{1}{2} \rho_s u_s^2
\]
Two-Fluid Plasma Model

Maxwell’s equations are used to evolve the electric and magnetic fields.

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0
\]
\[
\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_s \frac{q_s}{m_s} \rho_s \mathbf{u}_s
\]

Perfectly hyperbolic Maxwell’s equations\(^1\) are used to maintain divergence constraints on \(\mathbf{B}\) and \(\mathbf{E}\).

\(^1\)Dedner et al, Journal of Computational Physics, 2001
Hall-MHD Equation System by Applying Asymptotic Approximations

- Fluid equations for ions and Faraday’s law same as two-fluid model.
- Charge neutrality assumption eliminates electron continuity
- $\frac{m_e}{m_i} \to 0$ assumption reduces electron momentum to generalized Ohm’s law

\[ nq_e E = \nabla p_e - J_e \times B \]

- $c \to \infty$ assumption reduces Ampere’s law to

\[ J = \frac{1}{\mu_0} \nabla \times B \] where \( J = J_i + J_e \)

- Hall-MHD model incorporates electron energy equation
WARPX (Washington Approximate Riemann Plasma) is a general geometry 3-D finite volume and finite element code.

- Explicit finite element implementation: Runge-Kutta, discontinuous-Galerkin (RKDG) method
- Implicit finite element implementation: $\theta$-method
  - $\theta = 0$: Backward Difference Euler with discontinuous Galerkin
  - $\theta = \frac{1}{2}$: 2\textsuperscript{nd} order Crank-Nicolson with discontinuous Galerkin
- Same finite element spatial discretization for explicit and implicit time-stepping
- ICE Cluster at the University of Washington
Discontinuous Galerkin (DG) Method

- Hyperbolic balance law

\[
\frac{\partial Q}{\partial t} = -\nabla \cdot F + S,
\]

where \( Q \) is conserved variable, \( F \) is flux, \( S \) is source
- Multiply by basis and integrate over element, \( r \in [0, p) \)

\[
\int_{\Omega} \frac{\partial Q}{\partial t} v_r dV = -\int_{\Omega} (\nabla \cdot F) v_r dV + \int_{\Omega} S v_r dV
\]

\[= R_p \]

- Use integration by parts on the flux term to get:

\[
\int_{\Omega} (\nabla \cdot F) v_r dV = \int_{\partial \Omega} (F \cdot n) v_r d\Gamma - \int_{\Omega} F \cdot (\nabla v_r) dV
\]
Explicit Runge-Kutta Discontinuous Galerkin Method

- Riemann problem solved at each interface for edge fluxes (surface integral term for flux)
- Discontinuous Galerkin (DG) method enforces flux continuity but not continuity of the variable at each cell interface (not $C^0$)
- 3rd order Runge-Kutta used for time integration - RKDG

---

Implicit Discontinuous Galerkin Method\textsuperscript{1}

\textbf{θ-method:}

\[
R_e(Q_h^{n+1}) = \frac{M}{\Delta t} Q_h^{n+1} + \theta R_p(Q_h^{n+1}) - \left( \frac{M}{\Delta t} Q_h^n - (1 - \theta) R_p(Q_h^n) \right) = 0
\]

\begin{itemize}
  \item \textbf{R}_p\text{ represents the spatially-dependent fluxes and sources computed identical to the explicit method}
  \item \textbf{R}_e\text{ represents the non-linear residual}
  \item \textbf{M}\text{ represents the mass matrix containing the expansion coefficients for the DG method}
\end{itemize}

\textsuperscript{1}Wang and Mavriplis, Journal of Computational Physics, 2007
Implicit-DG

- Solve system of $Ax = b$ for $x$ by inverting matrix $A$.
- Jacobian computed numerically using a sparse matrix method (block tri-diagonal in 1-D and block penta-diagonal in 2-D)
- PETSC’s SNES (Scalable Non-linear Equation Solvers) used to solve system using a line-search Newton method
- Preconditioners will be included in future work. Some potentials,
  - $p$-based preconditioner
  - physics-based preconditioner$^1$

$^1$L. Chacon, Physics of Plasmas, 2008
Explicit Vs. Implicit for Two-Fluid Plasma Model

- **Two-fluid model characteristic time-scales:**
  - \( \tau_e = \frac{L}{c_e} \): electron sound transit time
  - \( \tau_i = \frac{L}{c_i} \): ion sound transit time
  - \( \tau_c = \frac{L}{c} \): light transit time
  - \( \tau_{ci} = \frac{1}{\omega_{ci}} \): ion cyclotron time
  - \( \tau_{ce} = \frac{1}{\omega_{ce}} \): electron cyclotron time
  - \( \tau_{pi} = \frac{1}{\omega_{pi}} \): ion plasma time
  - \( \tau_{pe} = \frac{1}{\omega_{pe}} \): electron plasma time

- **Explicit time-step restriction for DG:** CFL \( \leq \frac{1}{(2p - 1)} \), where \( p \) is the spatial (polynomial) order. Need to resolve all above time-scales for stability.

- **Implicit time-step depends on desired accuracy alone.** Simulations can be run at ion time-scales.
In 2-D, fully implicit two-fluid model has $18 \times p \times p$ equations where $p$ is spatial order for DG (in 1-D, $18 \times p$ equations) - large matrix to invert.

So, Maxwell’s equations and electron Euler equations solved implicitly using time-step at ion time-scales.

Ion fluid equations solved explicitly at ion time-scales.

Faster than solving the entire two-fluid system implicitly.

Smaller matrix to invert.

Still allows for realistic electron mass and real light speed.
Normalized initial conditions are presented.

- Number density (ion & electron)
- Pressure (ion & electron)
- Z-direction magnetic field
- Parameter space: \( \frac{\delta_i}{L} = 1, \frac{r_{Li}}{L} = 0.4 \)
Comparing Artificial and Realistic Parameters for MHD Shock Problem, after 0.1 Alfven Transit Times ($t_A$)

Artificial parameters:
\[
\frac{m_i}{m_e} = 25, \quad \frac{c}{v_A} = 10^2
\]

Realistic parameters:
\[
\frac{m_i}{m_e} = 1836.2, \quad \frac{c}{v_A} = 10^4
\]

- Left: electron number density
- Right: Z-direction magnetic field

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Realistic Parameters after $0.1t_A$

Explicit- and Implicit-DG results presented using realistic parameters for both

- **Left:** ion number density
- **Right:** Z-direction magnetic field
Realistic Parameters - Motivation for Implicit Time-Stepping

- For same resolution (500 cells), $\frac{m_i}{m_e} = 1836.2$, $\frac{c}{v_A} = 10^4$
- Implicit scheme allows realistic parameters with less computational effort
- Implicit time-step restricted by ion characteristic speeds
- Explicit time-step restricted by electron plasma frequency and light speed

<table>
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<tr>
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<th>$\Delta t$ (s)</th>
<th>Total CPU time (s)</th>
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<tbody>
<tr>
<td>Explicit</td>
<td>$6 \times 10^{-6}$</td>
<td>1065.2</td>
</tr>
<tr>
<td>Implicit</td>
<td>$9.5 \times 10^{-4}$</td>
<td>612.7</td>
</tr>
</tbody>
</table>
2-D Explicit-DG Results - Magnetic Reconnection

- Two-fluid model uses $\frac{m_i}{m_e} = 25$ and $\frac{c}{V_A} = 10$

- Left: Two-fluid solution of ion density at $\omega_{ci}t = 20$
- Right: Reconnected flux as a function of $\omega_{ci}t$
2-D Explicit-DG Results - Axisymmetric Z-pincho Lower Hybrid Drift Instability, $\frac{m_i}{m_e} = 25$, $\frac{c}{V_A} = 16$, $\frac{R_{Li}}{R_p} \approx 3$

- Left: Two-fluid ion density initial condition
- Right: Two-fluid ion density after 1.25$t_A$
Simulations of the Z-pinch show the growth of short-wavelength modes in the solution.

This is a two-fluid effect, and the short-wavelength modes are suspected to be the lower hybrid drift instability.

LHDI may explain anomalous resistivity for cross-field transport that has been observed in experiments.

Stronger physics basis than Chodura resistivity
3-D Explicit-DG Z-pinch - Sausage mode with LHDI

\[ \frac{m_i}{m_e} = 25, \quad \frac{c}{V_A} = 16, \quad \frac{R_{Li}}{R_p} \approx 3 \]

- Two-fluid model solution after \( t = 0, 4t_A, 5t_A \)
- Note small-wavelength instabilities on top of single wavelength perturbation
3-D Explicit-DG Z-pincho - Kink mode with LHDI

- Two-fluid model solution after $t = 0, 4t_A, 5t_A$
- Note small-wavelength instabilities on top of single wavelength perturbation
3-D Hill’s Vortex Evolution, $\frac{m_i}{m_e} = 25$, $\frac{c}{V_A} = 50$, $s \approx 10$

Plots are of ion number density in $x$-$y$ plane for midplane in $z$.
No perturbation applied but note formation of small-wavelength instability.
Future Work

- Future work: use implicit-DG to reproduce following explicit-DG problems that were previously simulated using artificial mass ratios and speed of light:
  - Magnetic Reconnection
  - Z-pinch Lower Hybrid Drift Instability
  - Field Reversed Configuration - investigate 3-D stability using the two-fluid model
- Reproduce using realistic parameters, $\frac{m_i}{m_e} = 1836.2$, $\frac{c}{V_A} = 10^4$, to check for differences in the physics introduced by artificial parameters. Expected to verify two-fluid effects such as the small-scale lower-hybrid drift instability
- Investigate 3-D applications of the Z-pinch and FRC in experimental geometry using the two-fluid plasma model
Conclusions

- Implicit-DG is tested in 1-D with two-fluid electro-magnetic MHD shock problem and is compared to explicit-DG.
- The explicit-DG algorithm is robust when the electron mass is artificially increased and the speed of light is artificially reduced.
- Using realistic ion-to-electron mass ratio and realistic ratio of light speed to Alfven velocity, the implicit-DG algorithm uses less computational effort than explicit-DG.
- The implicit method can take time-steps on the order of the ion time-scales whereas the time-step of the explicit method is on the order of the speed of light and the electron plasma frequency.
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