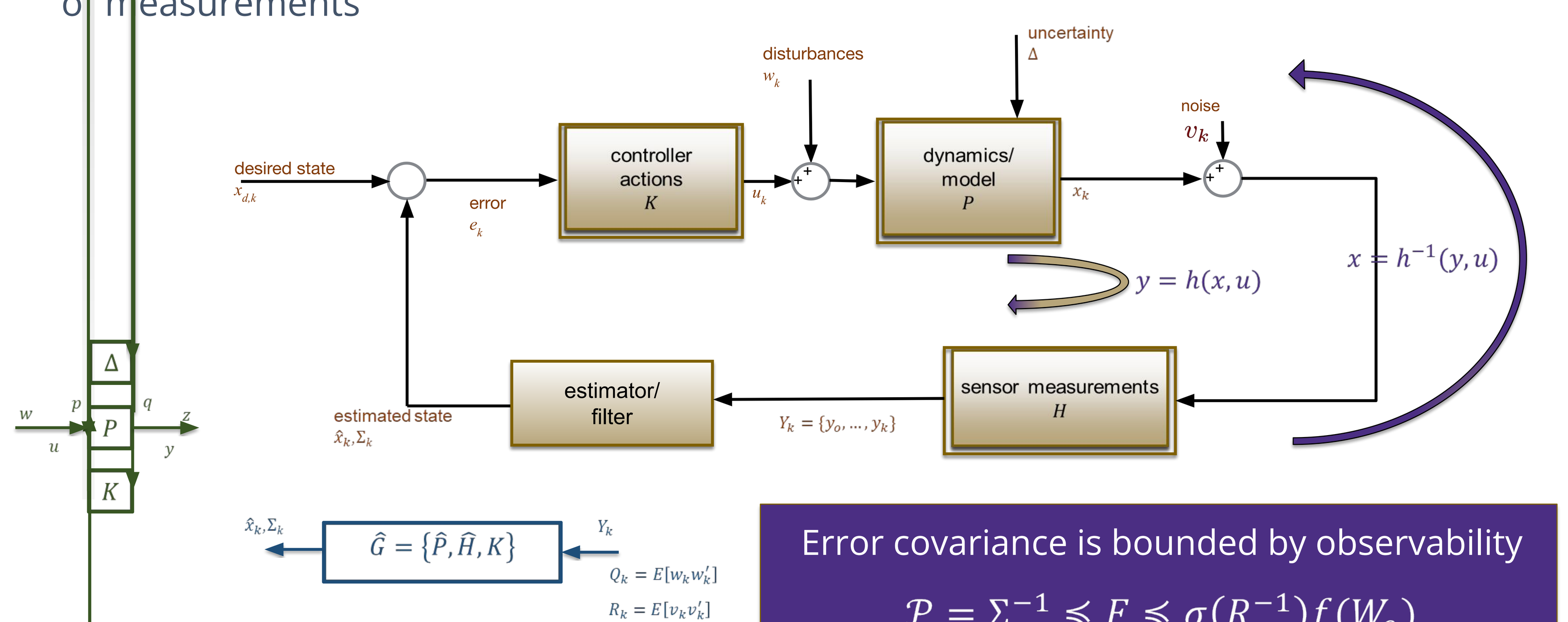
Sensor Placement Using Observability Gramians

STUDENTS: Ena Sundquist, Carey Whitehair, & Natalie Brace

What is observability & why observability Gramians?

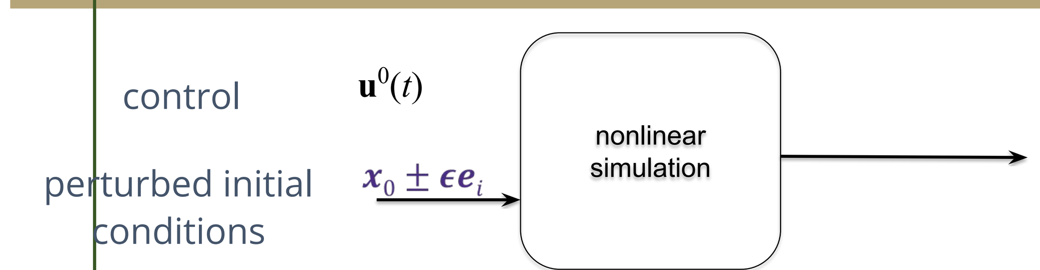
- **Observability** describes the ability to recreate a system's state from only knowledge of measurements



Error covariance is bounded by observability

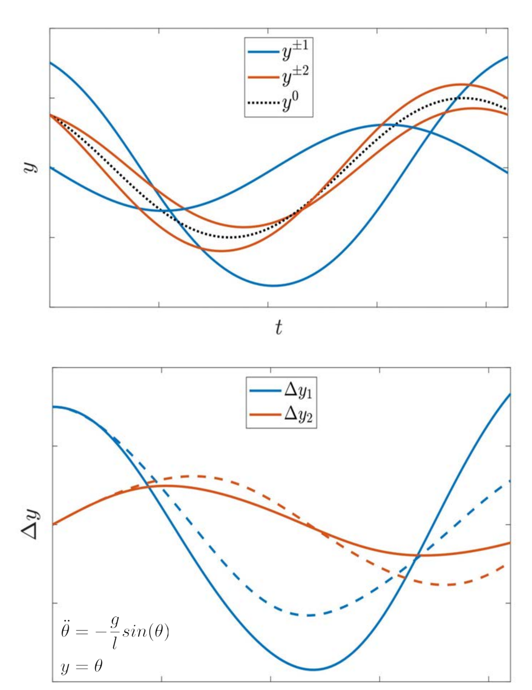
$$\mathcal{P} = \Sigma^{-1} \preccurlyeq F \preccurlyeq \underline{\sigma}(R^{-1}) f(W_o)$$

Empirical Observability Gramian



$$W_o^\epsilon = \frac{1}{4\epsilon^2}\int_{t_0}^{t_1}\begin{bmatrix}\Delta \mathbf{y}_1^\top(t)\\ \vdots \\ \Delta \mathbf{y}_n^\top(t)\end{bmatrix}\begin{bmatrix}\Delta \mathbf{y}_1 & \cdots & \Delta \mathbf{y}_n(t)\end{bmatrix} dt,$$

where $\Delta \mathbf{y}_i(t) = \mathbf{y}^{+i}(t) - \mathbf{y}^{-i}(t)$



Solid and dashed lines show results for perturbations introduced at times in the cycle that result in relatively high and low condition numbers

- The empirical observability Gramian is approximated by perturbing each of the initial conditions and simulating the results
- Does not require ability to analytically describe system

Measures of the Observability Gramian

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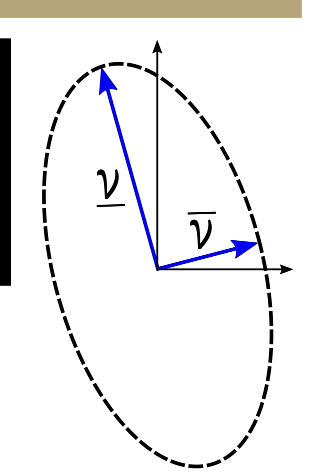

Level set of $||y||$

- Sensor placement results depend on which measures of the observability Gramian are included in the objective function for optimization
- The diagram shows how a large change in the least observable system mode is required to generate the same amount of output energy as a small change in the most observable system mode.

Nonlinear Observability

Control affine system:

$$\dot{\mathbf{x}} = \mathbf{f}(x,u) = \mathbf{f}_0(x) + \sum_{i=1}^{q} \mathbf{f}_i(x) u_i$$

$$y = h(\mathbf{x})$$

Output derivatives with nonzero control:

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} h(\mathbf{x}) \\ \frac{\partial h}{\partial \mathbf{x}}\dot{\mathbf{x}} \\ \vdots \\ \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial}{\partial \mathbf{x}}\left(\cdots\left(\frac{\partial h}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)\right)\right)\dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{f}}^0 h(x) \\ L_{\mathbf{f}}^1 h(x) \\ \vdots \\ L_{\mathbf{f}}^n h(x) \end{bmatrix} \neq \begin{bmatrix} L_{f_0}^0 h(x) \\ L_{f_0}^1 h(x) \\ \vdots \\ L_{f_0}^n h(x) \end{bmatrix}$$

Lie derivative:

$$L_{\mathbf{f}_i} h = \frac{\partial h}{\partial \mathbf{x}} \mathbf{f}_i$$

- For nonlinear systems, observability is not a global system property and can depend on control actions or noise

Airplane Model: Nonlinear System

- Equations of motion representing planar motion with rotation about the y-axis (pitch):

$$\dot{\mathbf{x}}_{\text{aug}} = \underbrace{\begin{bmatrix} \mathbf{v}_{\text{gen}} \\ -\mathbf{M}^{-1}\mathbf{C}(\mathbf{v}_{\text{gen}})\mathbf{v}_{\text{gen}} \\ \mathbf{0}_{4\times 1} \end{bmatrix}}_{\mathbf{f}_0(\mathbf{x}_{\text{aug}})} + \underbrace{\begin{bmatrix} \mathbf{0}_{1\times 3} \\ \mathbf{M}^{-1} \\ \mathbf{0}_{4\times 3} \end{bmatrix}}_{\mathbf{f}_\mathbf{i}(\mathbf{x}_{\text{aug}})} \tau$$

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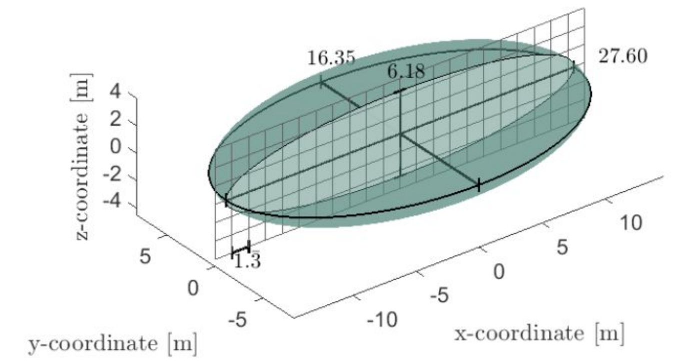
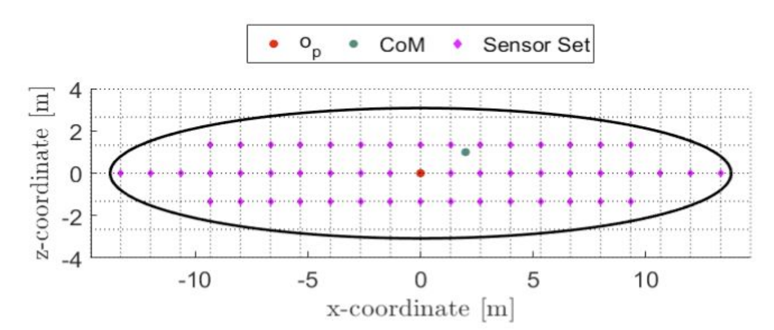
inertial parameters

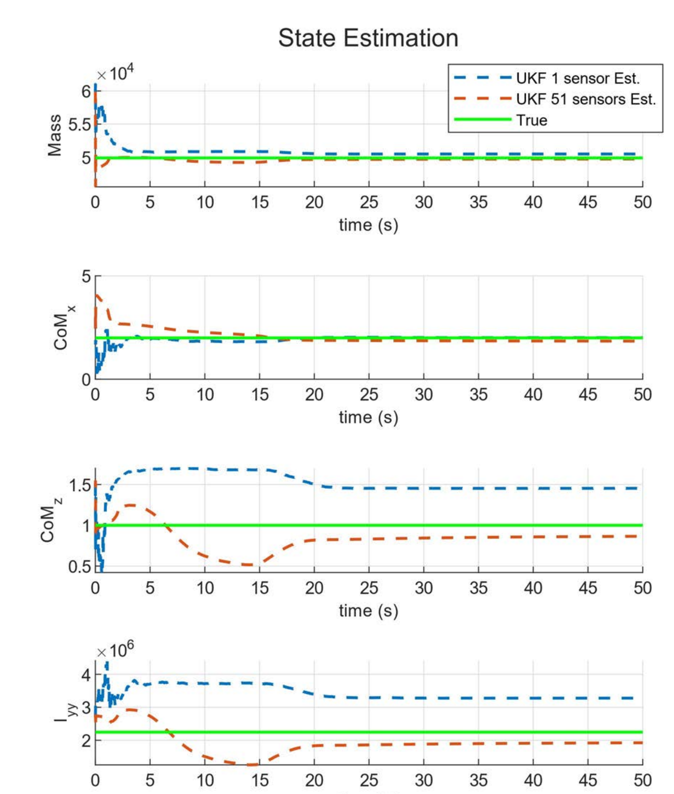


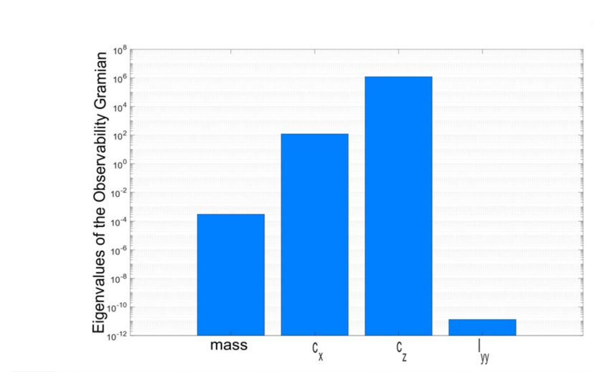
Rigid body ellipsoid

- Estimation of inertial parameters using a 6DoF IMU (accelerometer and gyroscope)
- Determine if parameters are observable using Lie algebra
- If observable, use empirical observability Gramian (EOG) to determine the degree of observability for each parameter
- Validate Lie algebra and EOG with UKF, which has noise included. This also informs estimation convergence time, and how this time varies with input force.



Cross section of rigid body ellipsoid showing geometric center, center of mass, and 51 sensor locations





Empirical observability Gramian eigenvalues with all 51 sensors active simulating steady level flight conditions

State estimation results comparing sensor data from a single sensor and fifty-one sensors (simultaneous)

E. Sundquist, C. Whitehair, and K. A. Morgansen, "Nonlinear Observability and Estimation of Rigid Body Inertial Parameters: a Multi-Sensor Strategy and its Experimental Validation," manuscript submitted to AIAA SciTech 2024 Forum.

Sensor Placement

- Determine where is the best place to locate sensors based on convex measures of the observability Gramian
- Introduce binary sensor selection variables α_ℓ
- Define the total observability Gramian as

$$\tilde{W}_o(\boldsymbol{\alpha}, t) = \sum_{\ell=1}^{n_p} W_o^\ell \alpha_\ell$$

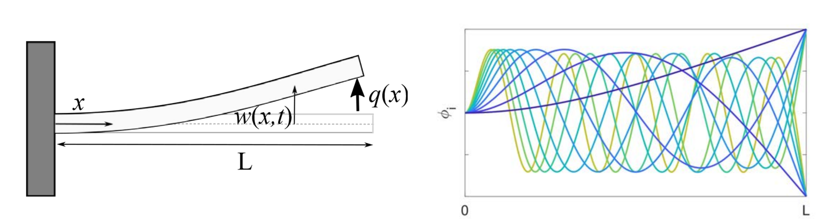
Convex optimization problem

$$\min_{\mathbf{a}} \quad J(\tilde{W}_o)$$

$$\text{subject to} \quad \sum_{\ell=1}^{n_p} a_\ell \leq p$$

$$0 \leq a_\ell \leq 1 \ \forall \ell$$

Cantilever Beam: Linear, Infinite Dimensional System



Cantilever beam and its first ten mode shapes.

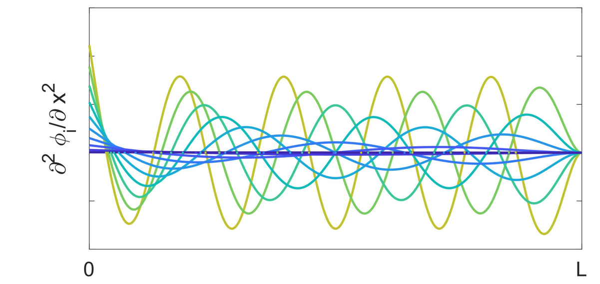
- Simple Euler-Bernoulli beam with strain measurements

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} = -\mu \frac{\partial^2 w(x,t)}{\partial t^2}$$

- Define mode shapes and modal coefficients

$$w(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t),$$

$$y(x_\ell, t) = h_\ell \sum_{i=1}^{\infty} \left.\frac{\partial^2 \phi(x)}{\partial x^2}\right|_{x=x_\ell} \eta_i(t)$$

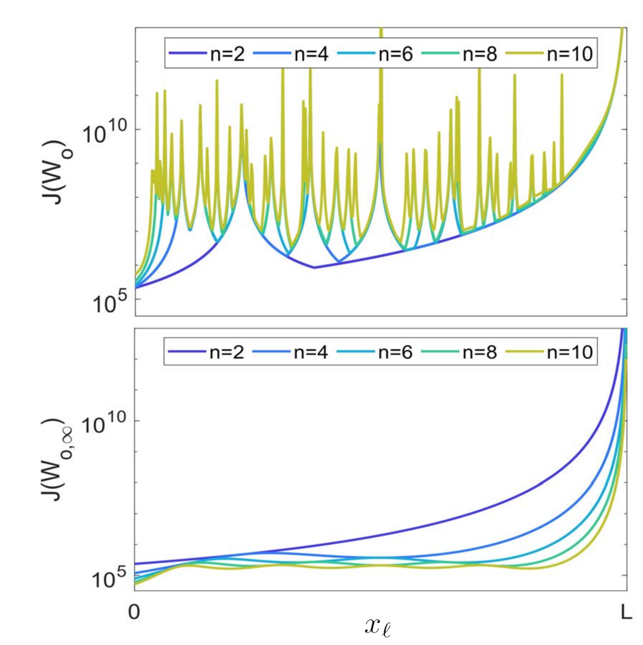


Second derivatives of the first ten mode shapes with respect to x (used to calculate strain)

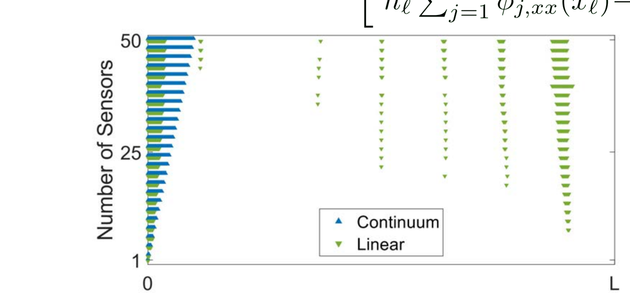
- Develop both continuum and empirical Gramians to verify methodology of perturbing the initial conditions of the infinite dimensional system using a function

$$W_\infty(t, x_\ell) = \int_0^t \delta W_\infty^\top \delta W_\infty d\tau$$

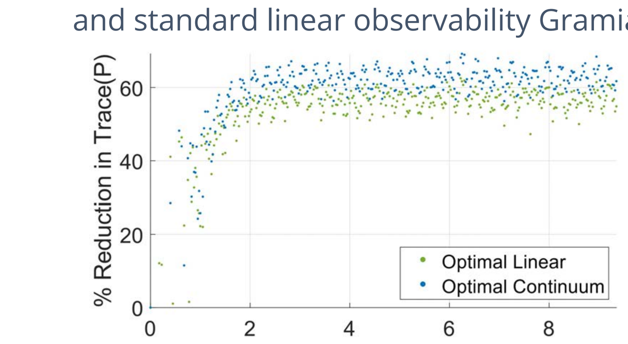
$$\delta W_\infty^\top = \begin{bmatrix} h_\ell \sum_{j=1}^{\infty} \phi_{j,xx}(x_\ell)\cos(\omega_j \tau) \\ h_\ell \sum_{j=1}^{\infty} \phi_{j,xx}(x_\ell)\frac{\sin(\omega_j \tau)}{\omega_j} \end{bmatrix}$$



Shapes of cost function $J(W) = \kappa(W) + w\lambda_{min}^{-1}(W)$ for the linear (top) and continuum (bottom) observability Gramians. Peaks in linear function correspond to zeros of $\partial^2\phi_i(x_\ell)/\partial x^2$.



Optimal sensor placement based on continuum and standard linear observability Gramians



Comparison of average estimated variance of optimally versus randomly placed sensors

N. L. Brace, N. B. Andrews, J. Upsal and K. A. Morgansen, "Sensor Placement on a Cantilever Beam Using Observability Gramians," 2022 IEEE 61st Conference on Decision and Control (CDC), Cancun, Mexico, 2022, pp. 388-395, doi: 10.1109/CDC51059.2022.9992639

Future Work | Acknowledgements

- Optimize placement for Using optimally placed sensors and validation with a physical experiment
- Extending the work on continuum systems to facilitate optimally placing sensors on more complex structures such as a flat plate and aircraft wing

Nicholas Andrews (graduate student) & Dr. Burak Boyacıoğlu (now post-doc)

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DEPARTMENT OF AERONAUTICS & ASTRONAUTICS

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