Theory of elastic solids reinforced by fibers that resist extension, flexure and twist

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Kirchhoff rod theory

Single fiber: $\mathbf{r}(S)$ - Position field

$$\mathbf{r}'(S) = \lambda \mathbf{d}_1, \quad \text{where} \quad \lambda = |\mathbf{r}'(S)| \qquad \mathbf{d}_i = \mathbf{A}(S)\mathbf{D}_i$$

$Orth^+$

Curvature and twist of the rod

$$\mathbf{d}_i' = \alpha \times \mathbf{d}_i$$

Axial vector

Curvature Frame-invariant formulation: strain-energy function $w(\lambda, \boldsymbol{\kappa})$

$$\boldsymbol{\kappa} = \mathbf{A}^t \alpha = \kappa_i \mathbf{D}_i \qquad \kappa_i = \tfrac{1}{2} e_{ijk} \mathbf{d}_k \cdot \mathbf{d}_j'$$

Equilibrium equations: $\mathbf{m}' + \varpi = \mathbf{f} \times \mathbf{r}'$ and $\mathbf{f}' + \mathbf{g} = \mathbf{0}$

$$\mathbf{m} = (\partial w / \partial \kappa_i)\mathbf{d}_i \qquad \mathbf{f} = \lambda^{-1}(\partial w / \partial \lambda)\mathbf{r}' + f_\alpha \mathbf{d}_\alpha$$

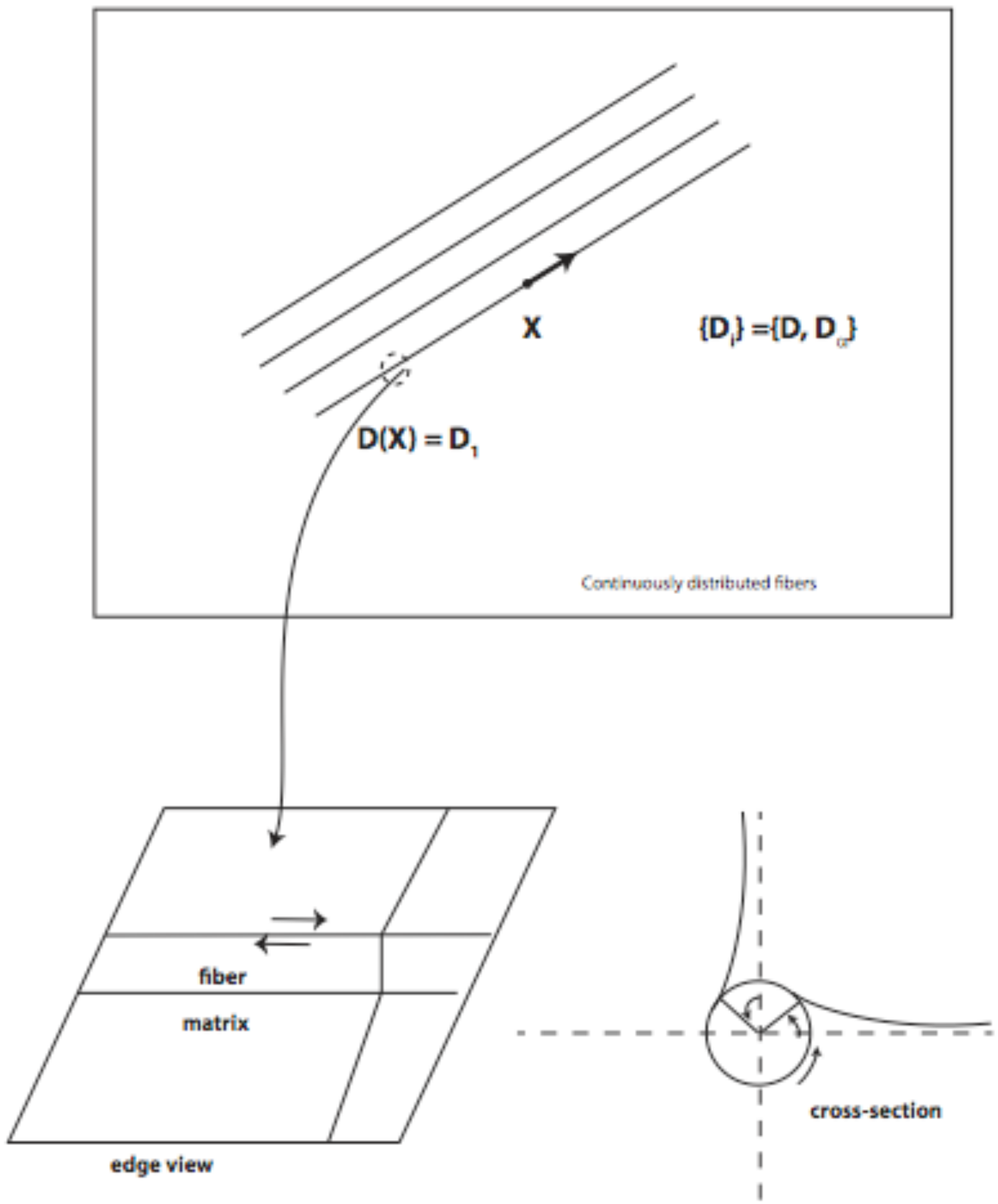
Kirchhoff rod theory

Strain-energy function most commonly used for isotropic rods of circular section

$$w(1, \kappa) = \tfrac{1}{2} GJ\kappa_1^2 + \tfrac{1}{2} EI\kappa_\alpha\kappa_\alpha$$

Then

$$\mathbf{m} = GJ\kappa_1\mathbf{d}_1 + EI\kappa_\alpha\mathbf{d}_\alpha = GJ\kappa_1\mathbf{d}_1 + EI\mathbf{d}_1 \times \mathbf{d}_1'$$



Fibers and matrix are kinematically independent; their interface convects as a material surface

Cosserat elasticity theory

Hadamard's compatibility condition requires that

$$\mathbf{F}^{+} - \mathbf{F}^{-} = \mathbf{f} \otimes \mathbf{N}$$

Deformation gradients in the fiber and the matrix at the interface

Unit normal to the interface

It follows that $\quad \mathbf{F}^{+}\mathbf{D} = \mathbf{F}^{-}\mathbf{D}, \quad$ but $\quad \mathbf{F}^{+}\mathbf{D}_{\alpha} \neq \mathbf{F}^{-}\mathbf{D}_{\alpha}$

If a fiber is sufficiently stiff relative to the matrix, its deformation gradient is given approximately by a rotation field $\mathbf{R}$

Thus $\quad \mathbf{F}\mathbf{D} = \lambda\mathbf{R}\mathbf{D} \quad$ where $\quad \lambda(= |\mathbf{F}\mathbf{D}|)$

Consider a referential energy density $\quad U(\mathbf{F}, \mathbf{R}, \mathbf{S}; \mathbf{X})$

Rotation gradient

$$\mathbf{F} = F_{iA}\mathbf{e}_i \otimes \mathbf{E}_A, \quad \mathbf{R} = R_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \quad \text{and} \quad \mathbf{S} = S_{iAB}\mathbf{e}_i \otimes \mathbf{E}_A \otimes \mathbf{E}_B$$

with

$$F_{iA} = \chi_{i,A} \quad \text{and} \quad S_{iAB} = R_{iA,B} \qquad \text{where} \qquad \begin{aligned} (\cdot)_{,A} &= \partial(\cdot)/\partial X_A \\ x_i &= \chi_i(X_A) \end{aligned}$$

Cosserat elasticity theory

The rotation field acts on the orthonormal triad field $\{\mathbf{D}_i(\mathbf{X})\}$

$$\{\mathbf{D}_i\} = \{\mathbf{D}, \mathbf{D}_\alpha\}$$

$\mathbf{D}(= \mathbf{D}_1)$ Unit tangent to a fiber in the reference configuration

$\mathbf{D}_\alpha$ Cross-sectional vectors embedded in the fiber

Thus $\mathbf{d}_i = \mathbf{R}\mathbf{D}_i$

The fiber is regarded as an embedded curve

$$\mathbf{F}\mathbf{D} = \lambda \mathbf{d}, \quad \text{where} \quad \mathbf{d} = \mathbf{R}\mathbf{D} \quad \text{and} \quad \lambda = |\mathbf{F}\mathbf{D}|$$

Constraints:

$$\mathbf{R}\mathbf{D}_\alpha \cdot \mathbf{F}\mathbf{D} = 0; \quad \alpha = 2, 3$$

Cosserat elasticity theory

Reduced strain-energy function

$$U(\mathbf{F}, \mathbf{R}, \mathbf{S}; \mathbf{X}) = U(\mathbf{QF}, \mathbf{QR}, \mathbf{QS}; \mathbf{X})$$

The restriction

$$U(\mathbf{F}, \mathbf{R}, \mathbf{S}; \mathbf{X}) = W(\mathbf{E}, \mathbf{\Gamma}; \mathbf{X})$$

where

$$\mathbf{E} = \mathbf{R}^t\mathbf{F} = E_{AB}\mathbf{E}_A \otimes \mathbf{E}_B; \quad E_{AB} = R_{iA}F_{iB}$$

$$\mathbf{\Gamma} = \Gamma_{DC}\mathbf{E}_D \otimes \mathbf{E}_C; \quad \Gamma_{DC} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}$$

The associated axial vectors

$$\gamma_{D(C)} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}$$

yielding

$$\mathbf{\Gamma} = \gamma_C \otimes \mathbf{E}_C$$

Cosserat elasticity theory

Stationary energy and equilibrium

The potential energy: $E = \int_{\xi} W dv - L$ Load potential

Consider the dead load problem, such that

$$L = \int_{\partial \xi_t} \mathbf{t} \cdot \chi da + \int_{\partial \xi_c} \mathbf{m}_i \cdot \mathbf{d}_i da$$

The virtual work of the force and couples is

$$\dot{L} = \int_{\partial \xi_t} \mathbf{t} \cdot \dot{\chi} da + \int_{\partial \xi_c} \mathbf{c} \cdot \omega da$$

where $\mathbf{c} = ax[(\mathbf{D}_i \otimes \mathbf{m}_i)\mathbf{R} - \mathbf{R}^t(\mathbf{m}_i \otimes \mathbf{D}_i)]$ $\omega = ax(\boldsymbol{\Omega})$

One parameter family: $\{\mathbf{F}(\mathbf{X};\epsilon), \mathbf{R}(\mathbf{X};\epsilon)\}$

Cosserat elasticity theory

Virtual-work statement

$$\int_{\xi} \dot{W} dv = \int_{\partial \xi_t} \mathbf{t} \cdot \dot{\chi} da + \int_{\partial \xi_c} \mathbf{c} \cdot \omega da$$

Global balance statements: Consider a rigid body motion

$$\chi(\mathbf{X};\epsilon) = \underset{}{\mathbf{Q}}(\epsilon)\chi_0(\mathbf{X}) + \mathbf{b}(\epsilon), \quad \mathbf{R}(\mathbf{X};\epsilon) = \underbrace{\mathbf{Q}}_{}{}^{\!\!\swarrow Orth^+}(\epsilon)\mathbf{R}_0(\mathbf{X}),$$

Strain energy invariance gives

$$\int_{\partial \xi} (\mathbf{t} \cdot \dot{\chi} + \mathbf{c} \cdot \omega) da = 0$$

where

$$\dot{\chi} = \mathbf{a} \times (\chi_0 - \mathbf{b}_0) + \dot{\mathbf{b}} \quad \text{and} \quad \omega = -\mathbf{R}_0^t \mathbf{a}$$

To obtain the second result we use

$$(\mathbf{R}^t \dot{\mathbf{R}})\mathbf{R}^t \mathbf{v} = \mathbf{R}^t (\mathbf{a} \times \mathbf{v}) = \mathbf{R}^t \mathbf{a} \times \mathbf{R}^t \mathbf{v}$$

Cosserat elasticity theory

Using $\mathbf{R}^t\dot{\mathbf{R}} = -\mathbf{\Omega}$ we conclude $\mathbf{a} = -\mathbf{R}\omega$

Thus
$$(\dot{\mathbf{b}} - \mathbf{a}\times\mathbf{b})\cdot\int_{\partial\xi}\mathbf{t}da + \mathbf{a}\cdot\int_{\partial\xi}(\chi\times\mathbf{t} - \mathbf{R}\mathbf{c})da = 0$$

yielding
$$\int_{\partial\xi}\mathbf{t}da = \mathbf{0} \quad \text{and} \quad \int_{\partial\xi}(\chi\times\mathbf{t} - \mathbf{R}\mathbf{c})da = \mathbf{0}$$

General case
$$\bar{E} = \int_{\xi}\bar{W}dv - L$$

$$\bar{W} = W + \Lambda_\alpha\mathbf{D}_\alpha\cdot\mathbf{E}\mathbf{D}$$

Lagrange multipliers

Cosserat elasticity theory

Virtual work $(\bar{E})^{\cdot} = 0$ reduces to

$$\int_{\xi}\{\dot{\Lambda}_{\alpha}\mathbf{D}_{\alpha}\cdot\mathbf{ED} + \omega\cdot(Div\mu + 2axSkw[(\sigma+\mathbf{\Lambda}\otimes\mathbf{D})\mathbf{E}^{t} + \mu\mathbf{\Gamma}^{t}]) - \dot{\chi}\cdot Div(\mathbf{R}\sigma + \lambda\otimes\mathbf{D})\}dv$$

$$= \int_{\partial\xi_t}\dot{\chi}\cdot[\mathbf{t} - (\mathbf{R}\sigma + \lambda\otimes\mathbf{D})\mathbf{n}]da + \int_{\partial\xi_c}\omega\cdot(\mathbf{c} + \mu\mathbf{n})da$$

where $\mathbf{\Lambda} = \Lambda_{\alpha}\mathbf{D}_{\alpha}, \quad \lambda = \mathbf{R\Lambda}, \quad \sigma = W_{\mathbf{E}}$ and $\mu = W_{\mathbf{\Gamma}}$

Hence the equilibrium equations:

$$Div(\mathbf{R}\sigma + \lambda\otimes\mathbf{D}) = \mathbf{0}, \quad Div\mu + ax\{2Skw[(\sigma+\mathbf{\Lambda}\otimes\mathbf{D})\mathbf{E}^{t} + \mu\mathbf{\Gamma}^{t}]\} = \mathbf{0} \quad \text{in} \quad \xi$$

and the boundary conditions

$$\mathbf{t} = (\mathbf{R}\sigma + \lambda\otimes\mathbf{D})\mathbf{n} \quad \text{on} \quad \partial\xi_t \quad \text{and} \quad \mathbf{c} + \mu\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\xi_c$$

Remarks

1. Fiber inextensibility is accommodated by appending the constraint $\mathbf{RD} \cdot \mathbf{FD} = 1$

$\boldsymbol{\Lambda}$ and λ are given by $\Lambda_i \mathbf{D}_i$ and $\Lambda_i \mathbf{d}_i$

Λ_1 is a kinematically undetermined density of axial force exerted on the fibers

2. Incompressibility entails the constraint $\det \mathbf{F} (= \det \mathbf{E}) = 1$

Accommodated by

$$\bar{W} = W + \Lambda_\alpha \mathbf{D}_\alpha \cdot \mathbf{ED} - p(\det \mathbf{E} - 1)$$

Relevant modified equations are

$$Div(\mathbf{R}\sigma - p\mathbf{F}^* + \lambda \otimes \mathbf{D}) = \mathbf{0} \quad \text{and} \quad \mathbf{t} = (\mathbf{R}\sigma - p\mathbf{F}^* + \lambda \otimes \mathbf{D})\mathbf{n}$$

3. The conventional theory of elasticity may be regarded as a special

$\mathbf{c}$ vanishes

W is independent of $\boldsymbol{\Gamma}$

$\mathbf{R}$ is constrained to be the rotation in the polar factorization of $\mathbf{F}$

Remarks

Then $\quad \mathbf{E} = \mathbf{U} \qquad \sigma \cdot \dot{\mathbf{U}} = \dot{W} = \mathbf{R}^t\mathbf{P} \cdot \dot{\mathbf{U}} \qquad$ where $\qquad \mathbf{P}(= \mathbf{F\Pi}) = W_{\mathbf{F}}$

$\sigma = Sym(\mathbf{R}^t\mathbf{P}) \qquad$ (Biot stress)

$Skw(\mathbf{R}^t\mathbf{F}) = \mathbf{0} \quad$ and $\quad \bar{W} = W + \mathbf{W} \cdot \mathbf{R}^t\mathbf{F}$

We obtain

$$(\bar{W})^{\cdot} = (\mathbf{P} + \mathbf{RW}) \cdot \dot{\mathbf{F}} + \mathbf{WU} \cdot \mathbf{\Omega} + \dot{\mathbf{W}} \cdot \mathbf{R}^t\mathbf{F}$$

The associated Euler equations are

$$Div(\mathbf{P} + \mathbf{RW}) = \mathbf{0} \quad \text{and} \quad Skw(\mathbf{WU}) = \mathbf{0}$$

A simple model for fiber-reinforced material

The kinematics of embedded fibers may be described in this framework by using

$$\kappa_i = \tfrac{1}{2} e_{ijk} \mathbf{D}_k \cdot \mathbf{R}^t \mathbf{R}' \mathbf{D}_j$$

Fibers are straight and untwisted $\quad \mathbf{D}_j' = \mathbf{0}$

Use $\quad R_{iA}' = R_{iA,B} D_B \quad$ to derive

$$\mathbf{R}^t \mathbf{R}' = R_{iC} S_{iAB} D_B \mathbf{E}_C \otimes \mathbf{E}_A = e_{ACD} \Gamma_{DB} D_B \mathbf{E}_C \otimes \mathbf{E}_A$$

$\Longrightarrow \quad \boldsymbol{\kappa} = \kappa_i \mathbf{D}_i$ is determined by $\boldsymbol{\Gamma}$

Thus the strain energy is described by a (different) constitutive function $W(\mathbf{E}, \boldsymbol{\kappa})$

To determine the associated response function $\boldsymbol{\mu}$

$$\dot{\kappa}_i = \mathbf{d}_i \cdot \mathbf{a}', \quad \text{where} \quad \mathbf{a} = ax(\dot{\mathbf{R}}\mathbf{R}^t)$$

$\dot{\kappa}_i = \mathbf{D}_i \cdot \mathbf{R}^t \mathbf{a}' = -\mathbf{D}_i \cdot \mathbf{R}^t (\mathbf{R}\omega)' \qquad$ yielding

$$\dot{\kappa}_i = (\mathbf{R}^t \mathbf{R}') \mathbf{D}_i \cdot \boldsymbol{\omega} - \omega_i', \quad \text{where} \quad \omega_i = \boldsymbol{\omega} \cdot \mathbf{D}_i$$

A simple model for fiber-reinforced material

$$\omega_i' = \omega_{i,A} D_A \qquad \dot{\mathbf{E}} = \mathbf{0} \qquad \Longrightarrow \qquad \dot{W} = \mathbf{M} \cdot \dot{\boldsymbol{\kappa}}$$

where $\mathbf{M} = M_i \mathbf{D}_i$ with $M_i = \partial W / \partial \kappa_i$

$$\dot{W} = \boldsymbol{\omega} \cdot [Div(\mathbf{M} \otimes \mathbf{D}) + (\mathbf{R}^t \mathbf{R}')\mathbf{M}] - Div[(\mathbf{M} \otimes \mathbf{D})^t \boldsymbol{\omega}]$$

$$\mu = \mathbf{M} \otimes \mathbf{D}$$

The moment-of-momentum balance specializes to

$$\mathbf{M}' + (\mathbf{R}^t \mathbf{R}')\mathbf{M} + ax\{2Skw[(\sigma + \boldsymbol{\Lambda} \otimes \mathbf{D})\mathbf{E}^t]\} = \mathbf{0}, \quad \text{where} \quad \mathbf{M}' = (\nabla \mathbf{M})\mathbf{D}$$

The associated boundary condition becomes $\mathbf{c} = -(\mathbf{D} \cdot \mathbf{n})\mathbf{M},$

The model may be recast in a form more easily recognizable from rod theory by introducing the field

$$\mathbf{m} = M_i \mathbf{d}_i = \mathbf{R}\mathbf{M}$$

This yields $\mathbf{M}' + (\mathbf{R}^t \mathbf{R}')\mathbf{M} = \mathbf{R}^t \mathbf{m}'$

A simple model for fiber-reinforced material

We observe that

$$ax\{2Skw[(\mathbf{\Lambda} \otimes \mathbf{D})\mathbf{E}^t]\} = ax[2Skw(\mathbf{R}^t\boldsymbol{\lambda} \otimes \mathbf{R}^t\boldsymbol{\chi}')], \quad \text{where} \quad \boldsymbol{\chi}' = \mathbf{FD}$$

Using the easily derived rule $\quad ax(\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}) = -\mathbf{a} \times \mathbf{b}$

We obtain $\quad ax[2Skw(\mathbf{R}^t\boldsymbol{\lambda} \otimes \mathbf{R}^t\boldsymbol{\chi}')] = \mathbf{R}^t\boldsymbol{\chi}' \times \mathbf{R}^t\boldsymbol{\lambda} = \mathbf{R}^t(\boldsymbol{\chi}' \times \boldsymbol{\lambda})$

and hence

$$\mathbf{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} + \mathbf{R}\{ax(\boldsymbol{\sigma}\mathbf{E}^t - \mathbf{E}\boldsymbol{\sigma}^t)\} = \mathbf{0}, \quad \text{where} \quad \mathbf{m}' = (\nabla\mathbf{m})\mathbf{D}$$

The linear-momentum balance may be recast as

$$\boldsymbol{\lambda}' + Div(\mathbf{R}\boldsymbol{\sigma} - p\mathbf{F}^*) = \mathbf{0}, \quad \text{where} \quad \boldsymbol{\lambda}' = (\nabla\boldsymbol{\lambda})\mathbf{D}$$

The contribution to the net moment from the embedded fibers reduces to

$$-\mathbf{Rc} = \mathbf{m}(\mathbf{D} \cdot \mathbf{n})$$

To the leading order

$$W(\mathbf{E}, \boldsymbol{\kappa}) = W(\mathbf{E}, \mathbf{0}) + \tfrac{1}{2}\boldsymbol{\kappa} \cdot \mathbf{K}(\mathbf{E})\boldsymbol{\kappa} \qquad \text{where} \qquad \mathbf{K}(\mathbf{E}) = W_{\boldsymbol{\kappa}\boldsymbol{\kappa}}(\mathbf{E}, \mathbf{0})$$

Material symmetry and transverse isotropy

Two references $\xi \xrightarrow[\mathbf{Y}(\mathbf{X})]{} \mu$ $\quad \mathbf{F}_\xi = \mathbf{F}_\mu \mathbf{H}, \quad$ where $\quad \mathbf{H} = \nabla \mathbf{Y}$

$\det \mathbf{H} = 1$ impose $\mathbf{Y}(\mathbf{X}_0) = \mathbf{X}_0$

$\mathbf{R}_\xi = \mathbf{R}_\mu \mathbf{L}$ where $\mathbf{L} = \mathbf{G}_i \otimes \mathbf{D}_i$

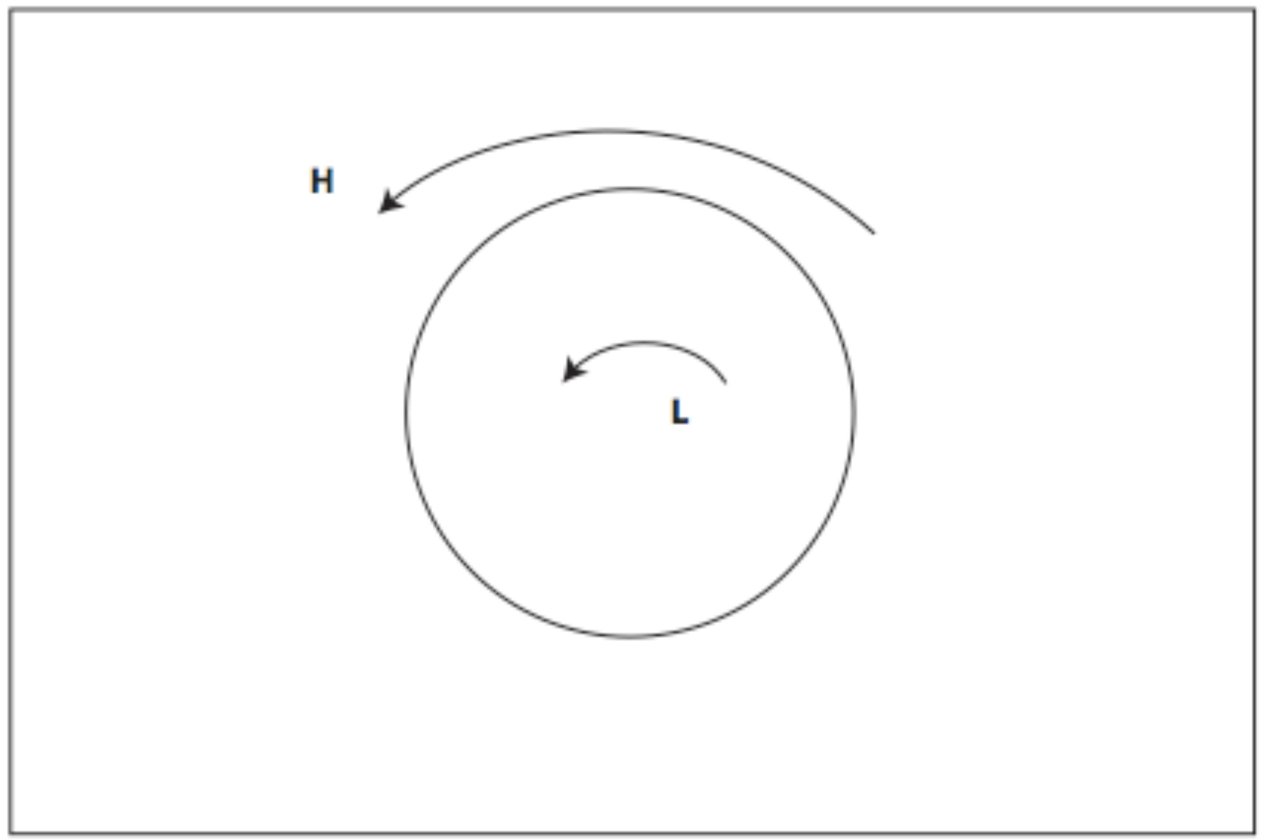
$\mathbf{D}$ is a material vector, thus impose $\mathbf{HD} = |\mathbf{HD}|\, \mathbf{LD}$

The rotation-gradient fields are related by

$$S^{(\xi)}_{iAB} = [S^{(\mu)}_{iCD} L_{CA} + R^{(\mu)}_{iC} L_{CA,D}] H_{DB}, \quad \text{where} \quad L_{CA,D} = \partial L_{CA} / \partial Y_D$$

Given $U_\xi(\mathbf{F}_\xi, \mathbf{R}_\xi, \mathbf{S}_\xi; \mathbf{X}_0)$

$$U_\mu(F^{(\mu)}_{iA}, R^{(\mu)}_{iA}, S^{(\mu)}_{iAB}; X^0_A) = U_\xi(F^{(\mu)}_{iB} H_{BA}, R^{(\mu)}_{iB} L_{BA}, [S^{(\mu)}_{iCD} L_{CA} + R^{(\mu)}_{iC} L_{CA,D}] H_{DB}; X^0_A)$$



Symmetry transformations for an isotropic fiber
embedded in an isotropic matrix

Material symmetry and transverse isotropy

Suppose now that the two references respond identically to given deformation and director rotation fields

$$U_\xi(\mathbf{F}, \mathbf{R}, \mathbf{S}; \mathbf{X}_0) = U_\mu(\mathbf{F}, \mathbf{R}, \mathbf{S}; \mathbf{X}_0)$$
$$U_\xi(F_{iA}, R_{iA}, S_{iAB}; X_A) = U_\xi(F_{iB}H_{BA}, R_{iB}L_{BA}, [S_{iCD}L_{CA} + R_{iC}L_{CA,D}]H_{DB}; X_A)$$

Confine attention to proper-orthogonal $\mathbf{H}$

We remove an inessential orientational degree of freedom in the local change of reference

$$\mathbf{D} = \mathbf{HD} = \mathbf{LD}$$

Application to the simplified model and specialization to transverse isotropy

Curvature twist vectors related by $\boldsymbol{\kappa}_\xi = \mathbf{L}^t \boldsymbol{\kappa}_\mu$

whereas $\mathbf{E}_\xi = \mathbf{L}^t \mathbf{E}_\mu \mathbf{H}$

The associated strain-energy functions satisfy $W_\xi(\mathbf{E}, \boldsymbol{\kappa}) = W_\mu(\mathbf{E}, \boldsymbol{\kappa})$

at the pivot point X_0 where $W_\mu(\mathbf{E}_\mu, \boldsymbol{\kappa}_\mu) = W_\xi(\mathbf{E}_\xi, \boldsymbol{\kappa}_\xi)$

Hence the restriction $W_\xi(\mathbf{E}, \boldsymbol{\kappa}) = W_\xi(\mathbf{L}^t \mathbf{E} \mathbf{H}, \mathbf{L}^t \boldsymbol{\kappa})$

Material symmetry and transverse isotropy

If the reinforced material is transversely isotropic, with the fibers perpendicular to the planes of isotropy, then this holds for all rotations

$$\mathbf{H}, \mathbf{L} \in S, \quad \text{where} \quad S = \{\mathbf{Q} \in Orth^{+} \quad \text{with} \quad \mathbf{QD} = \mathbf{D}\}.$$

For example, Kirchhoff's theory suggests strain-energy of the type

$$W(\mathbf{E}, \boldsymbol{\kappa}) = W_1(\mathbf{E}) + W_2(\mathbf{E})(\boldsymbol{\kappa} \cdot \mathbf{D})^2 + W_3(\mathbf{E}) \left|\mathbf{1}\boldsymbol{\kappa}\right|^2, \quad \text{with} \quad \mathbf{1} = \mathbf{I} - \mathbf{D} \otimes \mathbf{D},$$

Thus, we have $\quad \boldsymbol{\kappa} \cdot \mathbf{D} = \mathbf{L}^t \boldsymbol{\kappa} \cdot \mathbf{D} \quad \text{and} \quad |\mathbf{1}\boldsymbol{\kappa}| = \left|\mathbf{1}\mathbf{L}^t \boldsymbol{\kappa}\right| \quad \text{for all} \quad \mathbf{L} \in S$

Symmetry condition becomes $\quad W_i(\mathbf{E}) = W_i(\mathbf{L}^t \mathbf{E} \mathbf{H}); \quad i = 1, 2, 3.$

This is a non-standard representation problem

A list I of functionally independent scalars that satisfy the symmetry condition individually, for all $\mathbf{L}, \mathbf{H} \in S$

$$I = \{I_1, ..., I_9\},$$

Material symmetry and transverse isotropy

where

$$I_1 = tr(\mathbf{E}^t\mathbf{E}),\ I_2 = tr[(\mathbf{E}^t\mathbf{E})^2],\ I_3 = \det\mathbf{E},\ \ I_4 = \mathbf{D}\cdot\mathbf{E}\mathbf{D},\ I_5 = \mathbf{D}\cdot(\mathbf{E}^t\mathbf{E})\mathbf{D},$$
$$I_6 = \mathbf{D}\cdot(\mathbf{E}\mathbf{E}^t)\mathbf{D},\ I_7 = \mathbf{D}\cdot\mathbf{E}^*\mathbf{D},\ I_8 = \mathbf{D}\cdot(\mathbf{E}^t\mathbf{E})^2\mathbf{D},\ I_9 = \mathbf{D}\cdot(\mathbf{E}\mathbf{E}^t)^2\mathbf{D},$$

where $\quad \mathbf{E}^* = (\det\mathbf{E})\mathbf{E}^{-t}, \quad \det\mathbf{E} = \det\mathbf{F}, \quad \mathbf{E}^t\mathbf{E} = \mathbf{C}, \quad \mathbf{E}\mathbf{E}^t = \mathbf{R}^t\mathbf{B}\mathbf{R}$

with $\quad \mathbf{C} = \mathbf{F}^t\mathbf{F} \quad$ and $\quad \mathbf{B} = \mathbf{F}\mathbf{F}^t$ are the right and left Cauchy-Green deformation tensors.

The response function: $\quad \boldsymbol{\sigma} = W_{\mathbf{E}} = (W_1)_{\mathbf{E}} + (\boldsymbol{\kappa}\cdot\mathbf{D})^2(W_2)_{\mathbf{E}} + |\mathbf{1}\boldsymbol{\kappa}|^2\,(W_3)_{\mathbf{E}}$

with $\quad (W_i)_{\mathbf{E}} = \sum_j W_{ij}(I_j)_{\mathbf{E}}, \quad$ where $\quad W_{ij} = \partial W_i/\partial I_j$

$(I_j)_{\mathbf{E}}$ $\quad$ are obtained using the chain rule

$$(I_j)_{\mathbf{E}}\cdot\dot{\mathbf{E}} = \dot{I}_j$$

Material symmetry and transverse isotropy

Useful identities:

$$tr(\mathbf{AB}) = tr(\mathbf{BA}) = tr(\mathbf{B}^t\mathbf{A}^t) \quad \text{and} \quad \mathbf{A}\cdot\mathbf{BC} = \mathbf{AC}^t\cdot\mathbf{B} = \mathbf{B}^t\mathbf{A}\cdot\mathbf{C}$$

for arbitrary tensors $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Thus,

$$(I_1)_{\mathbf{E}} = 2\mathbf{E}, \quad (I_2)_{\mathbf{E}} = 4\mathbf{EC}, \quad (I_3)_{\mathbf{E}} = \mathbf{E}^*, \quad (I_4)_{\mathbf{E}} = \mathbf{D}\otimes\mathbf{D}, \quad (I_5)_{\mathbf{E}} = 2\mathbf{E}(\mathbf{D}\otimes\mathbf{D}),$$
$$(I_6)_{\mathbf{E}} = 2(\mathbf{D}\otimes\mathbf{D})\mathbf{E}, \quad (I_7)_{\mathbf{E}} = I_7\mathbf{E}^{-t} - I_3\mathbf{E}^{-t}(\mathbf{D}\otimes\mathbf{D})\mathbf{E}^{-t}, \quad (I_8)_{\mathbf{E}} = 2\mathbf{E}[(\mathbf{D}\otimes\mathbf{D})\mathbf{C} + \mathbf{C}(\mathbf{D}\otimes\mathbf{D})],$$
$$(I_9)_{\mathbf{E}} = 2[(\mathbf{D}\otimes\mathbf{D})\mathbf{EC} + \mathbf{EE}^t(\mathbf{D}\otimes\mathbf{D})\mathbf{E}].$$

By requiring, $(\boldsymbol{\kappa}\cdot\boldsymbol{D})_{\boldsymbol{\kappa}} = \mathbf{D}$ and $(|\mathbf{1}\boldsymbol{\kappa}|^2)_{\boldsymbol{\kappa}} = 2\mathbf{1}\boldsymbol{\kappa}$.

The response function $\mathbf{M} = W_{\boldsymbol{\kappa}} = 2W_2(\mathbf{E})(\boldsymbol{\kappa}\cdot\mathbf{D})\mathbf{D} + 2W_3(\mathbf{E})\mathbf{1}\boldsymbol{\kappa}$ and

$$\mathbf{m} = 2W_2(\mathbf{E})(\boldsymbol{\kappa}\cdot\mathbf{D})\mathbf{d} + 2W_3(\mathbf{E})\kappa_\alpha\mathbf{d}_\alpha, \quad \text{where} \quad \kappa_\alpha\mathbf{d}_\alpha = \mathbf{d}\times\mathbf{d}' \quad \text{with} \quad \mathbf{d}' = (\nabla\mathbf{d})\mathbf{D},$$

We impose $W_{2,3} > 0, \Rightarrow$ the tensor $\mathbf{K}(\mathbf{E})$ is positive definite

The Legendre-Hadamard condition

Consider $$E = \int_{\kappa} W(\mathbf{E}, \boldsymbol{\kappa}) dv$$

The first variation: $$\dot{E} = \int_{\kappa} (W_{\mathbf{E}} \cdot \dot{\mathbf{E}} + W_{\boldsymbol{\kappa}} \cdot \dot{\boldsymbol{\kappa}}) dv$$

The second variation:

$$\ddot{E} = \int_{\kappa} (W_{\mathbf{E}} \cdot \ddot{\mathbf{E}} + W_{\boldsymbol{\kappa}} \cdot \ddot{\boldsymbol{\kappa}}) dv$$
$$+ \int_{\kappa} \{\dot{\mathbf{E}} \cdot W_{\mathbf{EE}}[\dot{\mathbf{E}}] + (W_{\mathbf{E}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}} \cdot \dot{\mathbf{E}} + (W_{\boldsymbol{\kappa}\mathbf{E}})\dot{\mathbf{E}} \cdot \dot{\boldsymbol{\kappa}} + \dot{\boldsymbol{\kappa}} \cdot (W_{\boldsymbol{\kappa}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}}\} dv.$$

Since $\dot{\mathbf{E}} = \boldsymbol{\Omega}\mathbf{R}^t\mathbf{F} + \mathbf{R}^t\nabla\mathbf{u}$, with $\mathbf{u} = \dot{\boldsymbol{\chi}}$, and $\boldsymbol{\Omega} = \dot{\mathbf{R}}^t\mathbf{R}$.

Then $\dot{\mathbf{E}} = \mathbf{R}^t(\nabla\mathbf{u} - \boldsymbol{\alpha}\mathbf{F})$, where $\boldsymbol{\alpha} = \dot{\mathbf{R}}\mathbf{R}^t$.

Further $\dot{\boldsymbol{\kappa}} = \dot{\kappa}_i \mathbf{D}_i$, where $\dot{\kappa}_i = \mathbf{d}_i \cdot \mathbf{a}'$ and $\mathbf{a} = ax\boldsymbol{\alpha}$

The latter yields $$\dot{\boldsymbol{\kappa}} = \mathbf{R}^t\mathbf{a}'$$

The Legendre-Hadamard condition

The second variations are

$\ddot{\boldsymbol{\kappa}} = \mathbf{R}^t\mathbf{b}' + \dot{\mathbf{R}}^t\mathbf{a}' = \mathbf{R}^t\mathbf{b}' - \mathbf{R}^t\boldsymbol{\alpha}\mathbf{R}\dot{\boldsymbol{\kappa}}$ where $\mathbf{b} = ax\boldsymbol{\beta}$ with $\boldsymbol{\beta} = \dot{\boldsymbol{\alpha}}$ and

$\ddot{\mathbf{E}} = \mathbf{R}^t(\nabla\mathbf{v} - \boldsymbol{\beta}\mathbf{F}) - \mathbf{R}^t\boldsymbol{\alpha}(\nabla\mathbf{u} + \mathbf{R}\dot{\mathbf{E}})$ where $\mathbf{v} = \ddot{\boldsymbol{\chi}}$

Accordingly

$$\begin{aligned}\int_\kappa (W_{\mathbf{E}} \cdot \ddot{\mathbf{E}} + W_{\boldsymbol{\kappa}} \cdot \ddot{\boldsymbol{\kappa}})dv &= \int_\kappa [W_{\mathbf{E}} \cdot \mathbf{R}^t(\nabla\mathbf{v} - \boldsymbol{\beta}\mathbf{F}) + W_{\boldsymbol{\kappa}} \cdot \mathbf{R}^t\mathbf{b}']dv \\ &- \int_\kappa [W_{\mathbf{E}} \cdot \mathbf{R}^t\boldsymbol{\alpha}(\nabla\mathbf{u} + \mathbf{R}\dot{\mathbf{E}}) + W_{\boldsymbol{\kappa}} \cdot (\mathbf{R}^t\boldsymbol{\alpha}\mathbf{R})\dot{\boldsymbol{\kappa}}]dv\end{aligned}$$

At equilibrium: $0 = \dot{E} = \int_\kappa [W_{\mathbf{E}} \cdot \mathbf{R}^t(\nabla\mathbf{u} - \boldsymbol{\alpha}\mathbf{F}) + W_{\boldsymbol{\kappa}} \cdot \mathbf{R}^t\mathbf{a}']dv$

for all $\mathbf{u}$ and $\mathbf{a}$ such that

$$0 = \mathbf{D}_\alpha \cdot \dot{\mathbf{E}}\mathbf{D} = \mathbf{D}_\alpha \cdot \mathbf{R}^t(\nabla\mathbf{u} - \boldsymbol{\alpha}\mathbf{F})\mathbf{D} = \mathbf{d}_\alpha \cdot (\mathbf{u}' - \mathbf{a} \times \boldsymbol{\chi}')$$

The Legendre-Hadamard condition

Admissible second variations satisfy

$$0 = \mathbf{D}_\alpha \cdot \ddot{\mathbf{E}}\mathbf{D} = \mathbf{D}_\alpha \cdot \mathbf{R}^t(\nabla\mathbf{v} - \boldsymbol{\beta}\mathbf{F})\mathbf{D} - \mathbf{D}_\alpha \cdot \mathbf{R}^t\boldsymbol{\alpha}(\nabla\mathbf{u} + \mathbf{R}\dot{\mathbf{E}})\mathbf{D}$$
$$= \mathbf{d}_\alpha \cdot (\mathbf{v}' - \mathbf{b} \times \boldsymbol{\chi}') - \mathbf{d}_\alpha \cdot \mathbf{a} \times (\mathbf{u}' + \mathbf{R}\dot{\mathbf{E}}\mathbf{D}).$$

Second variations satisfying $\mathbf{d}_\alpha \cdot (\mathbf{v}' - \mathbf{b} \times \boldsymbol{\chi}') = 0.$ become

$$\ddot{E} = \int_\kappa \{\dot{\mathbf{E}} \cdot W_{\mathbf{EE}}[\dot{\mathbf{E}}] + (W_{\mathbf{E}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}} \cdot \dot{\mathbf{E}} + (W_{\boldsymbol{\kappa}\mathbf{E}})\dot{\mathbf{E}} \cdot \dot{\boldsymbol{\kappa}} + \dot{\boldsymbol{\kappa}} \cdot (W_{\boldsymbol{\kappa}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}}\}dv$$
$$- \int_\kappa [W_{\mathbf{E}} \cdot \mathbf{R}^t\boldsymbol{\alpha}(\nabla\mathbf{u} + \mathbf{R}\dot{\mathbf{E}}) + W_{\boldsymbol{\kappa}} \cdot (\mathbf{R}^t\boldsymbol{\alpha}\mathbf{R})\dot{\boldsymbol{\kappa}}]dv$$

Subject to $\mathbf{d}_\alpha \cdot \mathbf{a} \times (\mathbf{u}' + \mathbf{R}\dot{\mathbf{E}}\mathbf{D}) = 0$

Recall $\boldsymbol{\chi}' = \mathbf{F}\mathbf{D} = \lambda\mathbf{d}$ where λ is the fiber stretch.

The variational derivative yields $\mathbf{u}' = \dot{\lambda}\mathbf{d} + \mathbf{a} \times \boldsymbol{\chi}'$

The Legendre-Hadamard condition

We have $\dot{\mathbf{E}}\mathbf{D} = \mathbf{R}^t(\mathbf{u}' - \mathbf{a} \times \boldsymbol{\chi}')$

and hence $0 = \mathbf{d}_\alpha \cdot \mathbf{a}\times(2\dot{\lambda}\mathbf{d} + \mathbf{a} \times \boldsymbol{\chi}')$.

This requires $c \in \mathbb{R}$ such that $2\dot{\lambda}\mathbf{a} \times \mathbf{d} + \lambda\mathbf{a}\times(\mathbf{a} \times \mathbf{d}) = c\mathbf{d}$

Taking the inner product with $\mathbf{a} \times \mathbf{d}$ yields $\dot{\lambda} = 0$. That is

$$\mathbf{u}' = \mathbf{a} \times \boldsymbol{\chi}' \quad \text{and} \quad \mathbf{a}\times(\mathbf{a} \times \boldsymbol{\chi}') = c\mathbf{d}$$

Let $\mathbf{e} = \mathbf{a}/|\mathbf{a}|$

Using the identity $\boldsymbol{\chi}' = (\mathbf{e} \cdot \boldsymbol{\chi}')\mathbf{e} + \mathbf{e} \times (\boldsymbol{\chi}' \times \mathbf{e})$ we get

$$(c + \lambda|\mathbf{a}|^2)\mathbf{d} = \lambda|\mathbf{a}|^2(\mathbf{e} \cdot \mathbf{d})\mathbf{e}.$$

The possibilities are:

(i) $\mathbf{e} \cdot \mathbf{d} = 0$, and $c = -\lambda|\mathbf{a}|^2$, or (ii) $\mathbf{e} \cdot \mathbf{d} = \pm 1$ and $c = 0$

We conclude $\mathbf{a} \in Span\{\mathbf{d}\}$ or $\mathbf{a} \in Span\{\mathbf{d}_\alpha\}$

The Legendre-Hadamard condition

$$f(S) = \begin{array}{c} S - S_1, \quad S_1 \leq S \leq S_2 \\ -\frac{\theta}{1-\theta}(S - S_3), \quad S_2 < S \leq S_3 \\ 0, \quad S \in [0, L] \setminus (S_1, S_3), \end{array}$$

$$\int_{S_1}^{S_3} \dot{\boldsymbol{\kappa}} \cdot (W_{\boldsymbol{\kappa}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}} dS = O(\Delta) \quad \text{as} \quad \Delta \to 0$$

$$0 \leq \Delta^{-1}\ddot{E} = \int_{\Omega} \{\Delta^{-1} \int_{S_1}^{S_3} \dot{\boldsymbol{\kappa}} \cdot (W_{\boldsymbol{\kappa}\boldsymbol{\kappa}})\dot{\boldsymbol{\kappa}} dS + \Delta^{-1} o(\Delta)\} da$$

$$\int_{\Omega} \Delta^{-1} [\int_{S_1}^{S_3} \mathbf{a}' \cdot \mathbf{R}(W_{\boldsymbol{\kappa}\boldsymbol{\kappa}})\mathbf{R}^t \mathbf{a}' dS] da \geq 0$$

The Legendre-Hadamard condition

Recalling that $\mathbf{a}' \to f'\mathbf{g}$ we have

$$\int_{S_1}^{S_3} \mathbf{a}' \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{a}' dS \to \int_{S_1}^{S_2} \mathbf{g} \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{g} dS + \tfrac{\theta^2}{(1-\theta)^2} \int_{S_2}^{S_3} \mathbf{g} \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{g} dS$$

We obtain

$$\Delta^{-1} \int_{S_1}^{S_3} \mathbf{a}' \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{a}' dS \to \theta \left[\mathbf{g} \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{g}\right]_1 + (1-\theta)\tfrac{\theta^2}{(1-\theta)^2} \left[\mathbf{g} \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{g}\right]_2$$

where $[\cdot]_{1,2}$ are mean values in the intervals (S_1, S_2) and (S_2, S_3),

Hence, we conclude that

$$\Delta^{-1} \int_{S_1}^{S_3} \mathbf{a}' \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{a}' dS \to \tfrac{\theta}{1-\theta} \mathbf{g} \cdot \mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^t\mathbf{g}$$

which proves the claim.

The Legendre-Hadamard condition

Finally,

$$\frac{\theta}{1-\theta}\int_{\Omega}\mathbf{g}\cdot\mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^{t}\mathbf{g}da \geq 0,$$

and the arbitrariness of Ω yields

$$\mathbf{g}\cdot\mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^{t}\mathbf{g} \geq 0 \quad \text{at all} \quad \mathbf{X}\in\kappa$$

where

$$\mathbf{R}(W_{\boldsymbol{\kappa\kappa}})\mathbf{R}^{t} = (\partial^2 W/\partial\kappa_i\partial\kappa_j)\mathbf{d}_i\otimes\mathbf{d}_j$$

The constraints require

$$\mathbf{g}\in Span\{\mathbf{d}\} \quad \text{or} \quad \mathbf{g}\in Span\{\mathbf{d}_\alpha\}$$

Accordingly,

$$\partial^2 W/\partial\kappa_1^2 \geq 0 \quad \text{and} \quad (\partial^2 W/\partial\kappa_\alpha\partial\kappa_\beta) \quad \text{is positive definite.}$$

These are the Legendre-Hadamard necessary conditions for the present model.

Example: Torsion of a right circular cylinder

We use cylindrical coordinates in reference and current placement:

$$\mathbf{X} = r\mathbf{e}_r(\theta) + z\mathbf{k}$$

$$\chi(\mathbf{X}) = r\mathbf{e}_r(\phi) + z\mathbf{k}, \quad \text{where} \quad \phi = \theta + \tau z$$

where τ - the twist per unit length - is constant.

Deformation gradient: $\mathbf{F} = \mathbf{Q}[\mathbf{I} + r\tau\mathbf{e}_\theta(\theta) \otimes \mathbf{k}]$

where $\mathbf{Q} = \mathbf{e}_r(\phi) \otimes \mathbf{e}_r(\theta) + \mathbf{e}_\theta(\phi) \otimes \mathbf{e}_\theta(\theta) + \mathbf{k} \otimes \mathbf{k} \in Orth^+$

This is isochoric. Hence, we consider the incompressibility constraint to be operative.

$\mathbf{D} = \mathbf{k}$

The fibers are aligned with the axis of the cylinder in the reference placement.

Fiber derivative: $(\cdot)' = \partial(\cdot)/\partial z$

Then, $\lambda\mathbf{d} = \mathbf{F}\mathbf{k} = \mathbf{k} + r\tau\mathbf{e}_\theta(\phi); \quad \lambda = \sqrt{1 + r^2\tau^2}$

$\mathbf{d}$: The unit tangent, λ : The fiber stretch

Example: Torsion of a right circular cylinder

For fiber reinforced solids, we suppose

$$W_1(\mathbf{E}) = \tfrac{1}{2}\mu(I_1 - 3), \quad W_2(\mathbf{E}) = \tfrac{1}{2}T \quad \text{and} \quad W_3(\mathbf{E}) = \tfrac{1}{2}F$$

and obtain response functions

$$\boldsymbol{\sigma} = \mu\mathbf{E} \quad \text{and} \quad \mathbf{m} = T(\mathbf{k}\cdot\boldsymbol{\kappa})\mathbf{d} + F\mathbf{d}\times\mathbf{d}'$$

$$\mathbf{m}' + \lambda\mathbf{d}\times\boldsymbol{\lambda} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\lambda}' + \mu div\mathbf{B} = gradp$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^t = \mathbf{I} + r\tau[\mathbf{e}_\theta(\phi)\otimes\mathbf{k} + \mathbf{k}\otimes\mathbf{e}_\theta(\phi)] + r^2\tau^2\mathbf{e}_\theta(\phi)\otimes\mathbf{e}_\theta(\phi)$$

$$div\mathbf{B} = -r\tau^2\mathbf{e}_r(\phi)$$

$$p(r) = p_0 - \tfrac{1}{2}\mu\tau^2 r^2$$

Example: Torsion of a right circular cylinder

$$\mathbf{d} \times \mathbf{d}' = \lambda^{-2} r\tau^2 [r\tau\mathbf{k} - \mathbf{e}_\theta(\phi)]$$

$$\mathbf{m}' = \lambda^{-1} r\tau^2 (\lambda^{-1} F\tau - T\kappa_1)\mathbf{e}_r(\phi)$$

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda} \cdot \mathbf{d})\mathbf{d} + \mathbf{d} \times (\boldsymbol{\lambda} \times \mathbf{d}) = -\lambda^{-3} r\tau^2 (\lambda^{-1} F\tau - T\kappa_1)[r\tau\mathbf{k} - \mathbf{e}_\theta(\phi)]$$

$$\kappa_1 = \lambda^{-1}(F/T)\tau$$

$$\mathbf{m} = F\tau\mathbf{k}$$

Example: Torsion of a right circular cylinder

$$(\mathbf{R}\boldsymbol{\sigma} - p\mathbf{F}^*)\mathbf{e}_r(\theta) = \mathbf{0} \quad \text{at} \quad r = a$$

$$\mu\mathbf{B}\mathbf{e}_r(\phi) = p\mathbf{e}_r(\phi) \quad \text{at} \quad r = a$$

$$(\mathbf{R}\boldsymbol{\sigma})\mathbf{F}^t - p\mathbf{I} = \mu[\tfrac{1}{2}\tau^2(r^2 - a^2) - 1]\mathbf{I} + \mu\mathbf{B}$$

Example: Torsion of a right circular cylinder

The Overall Response

Traction on a cross section: $\mathbf{t} = [(\mathbf{R}\boldsymbol{\sigma})\mathbf{F}^t - p\mathbf{I}]\mathbf{k} = \frac{1}{2}\mu\tau^2(r^2 - a^2)\mathbf{k} + \mu r\tau\mathbf{e}_\theta(\phi)$

Resultant force: $\mathbf{f} = \int_0^{2\pi}\int_0^a \mathbf{t} r dr d\phi = f(\tau)\mathbf{k}$

where $f(\tau) = -\frac{1}{4}\pi a^4\mu\tau^2$

is a manifestation of the well-known normal-stress effect in nonlinear elasticity theory

Torque: $\boldsymbol{\rho} = \int_0^{2\pi}\int_0^a (\boldsymbol{\chi} \times \mathbf{t} + \mathbf{m}) r dr d\phi = \rho(\tau)\mathbf{k}$

where $\rho(\tau) = \pi a^2\tau(F + \frac{1}{2}\mu a^2)$

Example: Flexure of a rectangular block

We use Cartesian coordinates in the reference and polar coordinates in the current placement:

$$\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad \boldsymbol{\chi}(\mathbf{X}) = r\mathbf{e}_r(\theta) + z\mathbf{k}, \quad \text{where} \quad r = f(x) \quad \text{and} \quad \theta = g(y)$$

Deformation gradient: $\mathbf{F} = f'\mathbf{e}_r \otimes \mathbf{i} + fg'\mathbf{e}_\theta \otimes \mathbf{j} + \mathbf{k} \otimes \mathbf{k}$

yielding $1 = J = f(x)f'(x)g'(y)$

For incompressibility: $g = Cy \quad \text{and} \quad f = \sqrt{C^{-1}x + B}$

where B is a constant and $C > 0$

We consider two cases:

(a) $\mathbf{D} = \mathbf{i}$

$$\lambda\mathbf{d} = \mathbf{F}\mathbf{i} = f'(x)\mathbf{e}_r \qquad \text{yielding} \qquad \mathbf{d} = \mathbf{e}_r(\theta) \quad \text{and} \quad \lambda = f'(x)$$

Example: Flexure of a rectangular block

Then, $\mathbf{d}' = \mathbf{d}_{,x} = \mathbf{e}_\theta \theta_{,x},$ vanishes because θ is a function of y alone.

The constitutive equation: $\mathbf{m} = T\kappa \mathbf{e}_r$

We have $0 = \mathbf{d} \cdot \mathbf{m}' = \mathbf{e}_r \cdot \mathbf{m}_{,x} = T\kappa_{,x}$

Example: Flexure of a rectangular block

(b) $\mathbf{D} = \mathbf{j}$

The fibers are initially vertical.

$$\lambda \mathbf{d} = \mathbf{F}\mathbf{j} = f g' \mathbf{e}_\theta \quad \text{yielding} \quad \mathbf{d} = \mathbf{e}_\theta \quad \text{and} \quad \lambda = f g' = C\sqrt{C^{-1}x + B}$$

Then, $\mathbf{d}' = \mathbf{d}_{,y} = \mathbf{e}_\theta' g'(y) = -C\mathbf{e}_r$

The constitutive equation: $\mathbf{m} = T\kappa \mathbf{e}_\theta + FC\mathbf{k}$

Using $0 = \mathbf{d} \cdot \mathbf{m}' = \mathbf{e}_\theta \cdot \mathbf{m}_{,y} = T\kappa_{,y}$ we conclude $\kappa' (= \kappa_{,y})$ vanishes.

If no twisting couples are applied at the horizontal boundaries, then

$$\mathbf{m} = FC\mathbf{k}.$$

yielding $\mathbf{m}' = \mathbf{0}$ and $\boldsymbol{\lambda} = \mathbf{0}$,

Example: Bending, stretching and shearing of a block

First we deform the block by flexure to the configuration $\mathbf{x}_1 = \boldsymbol{\chi}_1(\mathbf{X})$,

Then the block is sheared to the configuration

$$\mathbf{x}_2 = \boldsymbol{\chi}_2(\mathbf{x}_1) = r\mathbf{e}_r(\theta) + \varsigma\mathbf{k}, \quad \text{where} \quad \varsigma = z + \beta\theta$$

with β a positive constant.

We obtain $\mathbf{F} = \mathbf{F}_2\mathbf{F}_1$ with $\mathbf{F}_1$ as before and $\mathbf{F}_2 = \mathbf{I} + \beta r^{-1}\mathbf{k} \otimes \mathbf{e}_\theta$

$\lambda\mathbf{d} = \mathbf{F}\mathbf{j} = C(r\mathbf{e}_\theta + \beta\mathbf{k})$ yielding $\lambda = C\sqrt{r^2 + \beta^2}$

$$\mathbf{d}' = \mathbf{d}_{,y} = \frac{-Cr}{\sqrt{r^2+\beta^2}}\mathbf{e}_r(\theta) \quad \text{and} \quad \mathbf{d} \times \mathbf{d}' = \frac{Cr^2}{r^2+\beta^2}(\mathbf{k} - \beta r^{-1}\mathbf{e}_\theta)$$

We obtain $\kappa'(= \kappa_{,y}) = 0$ which yields $T\kappa\mathbf{d}' + F\mathbf{d} \times \mathbf{d}'' = \lambda\boldsymbol{\lambda} \times \mathbf{d}$,

$$\boldsymbol{\lambda} \times \mathbf{d} = \lambda^{-1}\left(F\frac{C^2\beta r}{r^2+\beta^2} - T\frac{\kappa C r}{\sqrt{r^2+\beta^2}}\right)\mathbf{e}_r$$

Example: Bending, stretching and shearing of a block

This yields $\boldsymbol{\lambda} = \mathbf{d} \times (\boldsymbol{\lambda} \times \mathbf{d})$ in terms of $\kappa(x)$

A force-free solution $(\boldsymbol{\lambda} = \mathbf{0})$ with fiber twist is given by

$$\kappa(x) = \tfrac{F}{T} \frac{C\beta}{\sqrt{r^2+\beta^2}}, \quad \text{where} \quad r = f(x)$$

Thank You