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THE VORTEX AS A CLOCK

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ABSTRACT

Using heuristic arguments, the fundamental effect of acceleration on dissipation in self-similar turbulence is explored. If the ratio of the next vortex rotation period to the last one is always constant, a flow is temporally self-similar. This implies that the vortex rotation period is a linear function of time. For ordinary, unforced turbulence, the period increases linearly in time. By imposing an external time scale on the flow that decreases linearly in time, the dissipation rate is changed from that of the corresponding unforced flow. The dissipation rate depends on the time rate of change of the rotation period as well as the dimensions of the dynamic quantity controlling the flow.

1. INTRODUCTION

Among his many other achievements, Professor Roddam Narasimha pioneered the subject of forced, accelerating turbulence. He explored the relaminarization of a turbulent boundary layer due to strong acceleration and the reduction in spreading angle of a buoyant jet due to heat addition [1, 2]. Acceleration seem to inhibit entrainment and dissipation in general. This note pursues the effect of acceleration on dissipation in self-similar turbulence.

The common explanation for the constant spreading angle of jets and shear layers is self-similarity in space. Since there is no special, distinguished length scale, the local width of the flow must be proportional to the distance from the origin.

There is also an analogous self-similarity in time. If there is no special time scale, a turbulent flow is temporally self-similar. Because vorticity has dimensions of inverse time, all vortex properties, such as dissipation, should be intimately related to the temporal evolution of the vortex rotation period.

2. UNFORCED TURBULENCE

Unforced turbulence has no imposed time scale. In the absence of any imposed time scale, it follows that the vortex rotation period must be a linear function of the chronological age of the vortex. Otherwise, there would be a "radius of curvature" in the relationship between the vortex rotation period τ_v and time t. Such a radius of curvature implies a special time scale, in contradiction to the original assumption.

There are many examples of this linear relationship for $\tau_v(t)$ in unforced turbulence, i.e. when no external time scale is imposed on the flow. An examination of the large-scale structure in all the canonical laboratory flows quickly demonstrates that the largest eddies exhibit a vortex rotation period proportional to their age [3]. Even the boundary layer seems to approach this in the limit of infinite Reynolds number.

Another example in unforced turbulence is the vortex evolution following the energy cascade in the inertial subrange [4]. For large Reynolds number Re_δ, based on the largest eddy δ, there is a wide range of eddies larger than the Kolmogorov microscale λ_0 but smaller than δ. Kinetic energy is transferred from one eddy to the next by inertial forces without appreciable energy loss in the inertial subrange. The kinetic energy flux, e, is constant so that

$$e = \frac{v_\lambda^3}{\lambda} = \frac{\Delta U^3}{\delta} \tag{1}$$

Within the inertial subrange, an eddy of size λ and speed v_λ presumably hands over a large fraction of its energy in one turnover time given by

$$\tau_\lambda = \frac{\lambda}{v_\lambda} = \frac{\lambda^{\frac{2}{3}}}{e^{\frac{1}{3}}} = \left(\frac{\lambda}{\delta}\right)^{\frac{2}{3}} \tau_\delta, \tag{2}$$

where $\tau_\delta = \frac{\delta}{\Delta U}$ is the large scale vortex rotation period. A different question is how long it takes to reach scale λ from the start of the cascade. It has been suggested that self-similarity requires

$$dt = const \frac{\lambda^{\frac{2}{3}} d\lambda}{e^{\frac{1}{3}} \lambda}, \tag{3}$$

where dt is the differential time interval during which energy is transferred from scale λ to $\lambda - d\lambda$. Integration of (3) and the use of (1) provides the elapsed time $t(\lambda)$ to reach the scale λ starting from scale δ or

$$t(\lambda) = \left[1 - \left(\frac{\lambda}{\delta}\right)^{\frac{2}{3}}\right]\tau_\delta. \tag{4}$$

By combining equations (2) and (4),

$$\frac{\tau_\lambda}{\tau_\delta} = 1 - \frac{t(\lambda)}{\tau_\delta}. \tag{5}$$

The rotation period at scale λ is a linearly decreasing function of the elapsed time since the start of the cascade. Note that in the energy cascade, the dissipation rate is zero.

3. FORCED TURBULENCE

When an external time scale is imposed on a flow such that the vortex rotation period becomes equal to the imposed time scale, the flow is "forced". In comparison to the unforced case, the vortex evolution is consequently altered, in general destroying the self-similarity. However, if the imposed time scale itself is linear in time, self-similarity is preserved, yielding a new class of self-similar, forced flows.

The proof is simple [5]. Temporal self-similarity means no distinguished time scale. Therefore, the current vortex rotation period $\tau_v(t)$ at any time t is always a constant fraction of the next rotation period, $\tau_v(t+\tau_v(t))$. Expressed another way, the decrease in rotation period per rotation is a constant, β, independent of time.

$$\frac{\tau_v(t) - \tau_v(t+\tau_v(t))}{\tau_v(t)} = \beta. \tag{6}$$

Suppose $\tau_v(t)$ is an arbitrary polynomial of order n given by

$$\tau_v(t) = \tau_v(0) + a_1 t + a_2 t^2 + \cdots + a_n t^n. \tag{7}$$

Equations (6) and (7) yield

$$\frac{-a_1(\tau_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + O(t^{n^2})}{\tau_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n} = \beta. \tag{8}$$

The left hand side has terms of order t raised to the $(n^2 - n)$ power while the right-hand-side is independent of t. This can only be satisfied if $n^2 - n = 0$. The solutions are $n = 0$ or $n = 1$. Thus $a_i = 0$ for all

$i > 1$, and $a_1 = -\beta$. For this to be true, the only possible self-similar form is

$$\tau_v(t) = \tau_v(0) - \beta t, \tag{9}$$

where $\tau_v(0)$ is the rotation period at $t = 0$. In dimensionless form,

$$\frac{\tau_v(t)}{\tau_v(0)} = 1 - \frac{\beta t}{\tau_v(0)}. \tag{10}$$

From a comparison between equations (4) and (10), evidently $\beta = 1$ following the loss-free energy cascade in unforced turbulence.

Figure 1 illustrates in graphical form the self-similar evolution for $\beta > 0$. At the end of the first rotation ($n = 1$), the rotation period is reduced by a constant factor $(1 - \beta)$. At the end of the second rotation ($n = 2$), τ_v is again reduced by the same factor, and so on. At finite time $\frac{t}{\tau_v(0)} = \frac{1}{\beta}$, τ_v vanishes and the flow is singular.

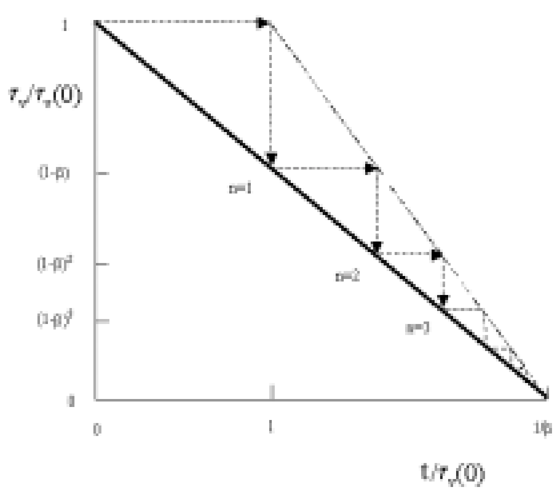

Figure 1: Temporal evolution of the vortex rotation period τ_v for $\beta > 0$ in self-similar flow.

To an observer following the cascading energy in the inertial sub-range, $\beta = 1$. However, to an observer following the large scale vortices in the same unforced flow, $\beta < 0$ [3–5]. The unforced, large-eddy rotation period is proportional to the vortex chronological age, since there is no other time scale available. The value of β thus depends on the observer. There seem to be two different canonical coordinate frames, in analogy with Eulerian and Lagrangian viewpoints of fluid motion, that yield different values of β for the same flow. An observer may either remain at a fixed eddy scale or follow the energy. In ordinary, unforced turbulence, the largest eddies extract energy from the mean flow via instabilities and hand it off to progressively smaller eddies in a cascade through the inertial subrange. Energy flows through the largest eddies. An observer remaining at the largest vortices could not follow the energy in unforced turbulence.

The reference frame of the observer can be defined in terms of the dimensionless flux of energy past him. Define γ to be the rate of energy flux e' past the observer normalized by the energy cascade flux e through the largest eddies so that

$$\gamma = \frac{e'}{e} = \frac{e'\delta}{\Delta U^3}. \tag{11}$$

For an observer fixed at scale $\lambda = \delta$, $\gamma = 1$ and $\beta < 0$ for unforced turbulence. For an observer moving down the cascade with the energy flux, $\gamma = 0$ and $\beta = 1$. This implies that the cascade process is interrupted, shutting off the flow of energy from the largest eddies into the cascade.

4. VORTICITY

The self-similar evolution represented by equation (10) is also the solution to the scalar equation

$$\frac{D\omega}{Dt} = \beta\omega^2, \tag{12}$$

if $\omega = \frac{1}{\tau_v}$. The vector vorticity equation for constant density flow in the absence of body forces is

$$\frac{D\boldsymbol{\omega}}{Dt} = (\nabla\mathbf{u})\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}. \tag{13}$$

This suggests that, for inviscid self-similar flow, the magnitude of the nonlinear vortex stretching term $(\nabla\mathbf{u})\boldsymbol{\omega}$ corresponds to $\beta\omega^2$. Thus, there is a kind of symmetry between the rate of deformation tensor and the vorticity as noted by G. Balle (private communication), such that their inner product is always proportional to the square of the magnitude of the vorticity. The rate of deformation becomes proportional to the vorticity magnitude, and the proportionality coefficient is simply β.

5. DISSIPATION RATE

Every canonical, unforced turbulent flow has a certain conserved quantity Q that constrols the dynamics. For example, in the shear layer it is the velocity jump ΔU, and in the jet it is the thrust per unit mass. The large eddies evolve so that their period $\tau_v(t)$ is proportional to t, and their size $\delta(t)$ is then determined by the further constraint of the conserved quantity.

In a forced flow, it is proposed that the turbulent dissipation rate is determined by the acceleration parameter β and the dimensions of the dynamic quantity Q. Take the dimensions of Q to be $\frac{(length)^m}{(time)^n}$. It is assumed that the controlling quantity is of super-exponential form, thereby imposing a linear e-folding time scale $(\tau_0 - \beta t)$ on the flow.

$$Q = Q_0 e^{\frac{t}{(\tau_0 - \beta t)}}. \tag{14}$$

The dissipation rate per unit mass has dimensions $\frac{(length)^2}{(time)^3}$. From dimensional considerations, the dissipation rate is proportional to $Q^{\frac{2}{m}} \tau^{-(3-\frac{2n}{m})}$.

The dimensionless dissipation rate D is definded to be the dissipation rate normalized by the rate of the corresponding unforced flow. The quantity $\alpha \equiv 3 - \frac{2n}{m}$ represents the amount by which the length scale in the dynamic quantity Q changes for a given change in the time scale. Because dissipation is associated with entrainment and hence changes in length scale, it seems a reasonable conjecture to take α as the natural scaling for the effect of β on dissipation. Consequently

$$\frac{dD}{D} = -\frac{d\beta}{\alpha} \tag{15}$$

so that

$$D = e^{-\frac{(\beta - \beta^*)}{\alpha}}, \tag{16}$$

where β^* is the value of β in the unforced flow. Equation (16) satisfies the requirement that the dissipation rate must be non-negative for any possible values of α and β. If α is positive, the turbulent dissipation rate is reduced as β increases. Table 1 lists several canonical laboratory flows, their dynamic quantity Q, and the corresponding values of m and n. For almost all cases, $\alpha > 0$, so that according to equation (16), the dissipation rate decreases as β increases. However, $\alpha = 0$ following the energy flux in the cascade, corresponding to zero dissipation rate. In Rayleigh-Taylor flow, $\alpha < 0$, so that the dissipation rate increases with increasing β.

flow	$[Q]=l^m/t^n$	m	n	$\alpha = (3-2n/m)$
shear layer	ΔU	1	1	1
round jet	T/ρ	4	2	2
plane jet	$T/\rho b$	3	2	5/3
round wake	θ^2	2	0	3
plane wake	θ	1	0	3
vortex ring	$w\delta^3$	4	1	5/2
vortex pair	$w\delta^2$	3	1	7/3
round thermal	F/ρ	4	2	2
plane thermal	$F/\rho b$	3	2	5/3
round plume	$g'w\delta^2$	4	3	3/2
plane plume	$g'w\delta^2/b$	3	3	1
inertial cascade	v_λ^3/λ	2	3	0
Rayleigh-Taylor	g'	1	2	-1

Table 1: Dimensions of the dynamical quantity Q for various flows. Note that $\alpha = 3 - \frac{2n}{m}$ is negative only for Rayleigh-Taylor flow.

6. RELATED EXPERIMENTS

6.1. Exponential jet

A flow with a constant imposed time scale is the exponential jet [6–10], which exhibits a constant vortex rotation period for the largest vortices and hence $\beta = 0$. Figure 2 illustrates the evolution of the large-scale rotation period as a function of time for all known self-similar flows. They are represented as a straight line on this figure, differing only in their slope, $-\beta$. Thus the exponential jet corresponds to a horizontal line. The unforced jet corresponds to an upward sloping line. From the spreading angle of the unforced jet, the slope is estimated to be approximately $-\beta^* = 0.4$. Taking $\alpha = 2$ for the round jet from table 1, the dissipation rate of the exponential jet should change by a factor of $D = e^{-0.2} = 0.82$ from the unforced flow according to equation (16). This is in accord with frame length measurements that show the exponential jet entrains and mixes at about a 20% lower rate than the unforced jet. Johari *et al.* note that acceleration is the only known means of affecting the mixing in the far-field jet [9, 10].

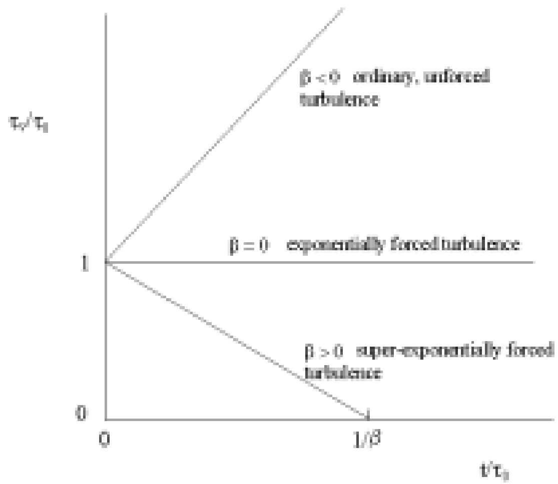

Figure 2: Vortex rotation period $\tau_v(t)$ for all self-similar flows.

6.2. Transverse exponential jet

In a transverse exponential jet, vortices are formed in the near field with an initial rotation period controlled by the nozzle time, the ratio of local nozzle diameter to speed [11]. However, as the vortices advect downstream, they are subjected to subsequent nozzles that increase in size and speed in an exponential way. The corresponding e-folding time introduces a second time scale. The ratio of these two times is a measure of the forcing. As this ratio increases above one, vortex rollup and entrainment are dramatically reduced.

6.3. Buoyant jet

Bhat and Narasimha [2] discovered that the spreading angle of an upward-moving turbulent jet was reduced when buoyancy was added. The evolution of the vortex rotation period $\tau_v(t)$ is changed by buoyancy addition in many natural flows, such as latent heat release in thunderstorms and vesciluation from vaporizing water in rising magma. If the buoyancy change is linear with height, the resulting super-exponential flow may be self-similar with extraordinarily low entrainment rates. Such a theory has been proposed to account for the geological puzzle that the second fluid intruded into a magma chamber is sometimes the first to erupt, little diluted by the first fluid [12].

6.4. Generalized Rayleigh-Taylor flow

It was originally thought that all forced flows would exhibit reduced entrainment and mixing with increasing β. Numerical simulations of a forced Rayleigh-Taylor flow first indicated that vortex sheet roll-up was inhibited by super-exponential acceleration. This should be reflected in an enhanced stability, as noted by J. Jimenez (private communication). However, a stability analysis by W. Criminale and L. Fisher (private communication) revealed that Rayleigh-Taylor flow would be more unstable as β increases. Subsequent numerical simulations by A. Cook and J. Greenough qualitatively confirmed the stability analysis, consistent with equation (16) and table 1.

The chaotic mixing regions associated with entrainment in accelerating turbulence have recently been investigated by Govindarajan [13]. The connection between the chaotic mixing regions and β has not yet been determined, but a prime suspect is the motion of the saddle point on the separatrix.

7. CONCLUSIONS

For all self-similar, turbulent flows, the vortex rotation period must be a linear function of time. The entrainment and dissipation rates depend on the slope of that linear function as well as the dimensions of the dynamical quantity of the flow. The evolution of the vortex rotation period acts as a clock, the speed and direction of which determine the dissipation rate.

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