Modeling open boundaries in dissipative MHD simulation

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\textsuperset{ABSTRACT}

The truncation of large physical domains to concentrate computational resources is necessary or desirable in simulating many natural and man-made plasma phenomena. Three open boundary condition (BC) methods for such domain truncation of dissipative magnetohydrodynamics (MHD) problems are described and compared here. A novel technique, lacuna-based open boundary conditions (LOBC), is presented for applying open BC to dissipative MHD and other hyperbolic and mixed hyperbolic-parabolic systems of partial differential equations. LOBC, based on manipulating Calderon-type near-boundary sources, essentially damp hyperbolic effects in an exterior region attached to the simulation domain and apply BC appropriate for the remaining parabolic effects (if present) at the exterior region boundary. Another technique, approximate Riemann BC (ARBC), is adapted from finite volume and discontinuous Galerkin methods. In ARBC, the value of incoming flux is specified using a local, characteristic-based method. A third commonly-used open BC, zero-normal derivative BC (ZND BC), is presented for comparison. These open BC are tested in several gas dynamics and dissipative MHD problems. LOBC are found to give stable, low-reflection solutions even in the presence of strong parabolic behavior, while ARBC are stable only when hyperbolic behavior is dominant. Pros and cons of the techniques are discussed and put into context within the body of open BC research to date.

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\textsuperset{1. Introduction}

In a variety of numerical problems, the computational domain must be a truncated approximation of the physical domain, allowing concentration of limited computational resources. An early example of such domain truncation is seen in the work of Charney et al. \cite{1}; in their numerical model of North American weather, the domain is cropped around North America and special attention is paid to boundary treatment. Boundary conditions (BC) that achieve domain truncation are variously referred to as open BC, artificial BC, non-reflecting BC, radiation BC, etc. In this manuscript, the term “open BC” will be used.

The primary objective of this research is to develop open BC methods for modeling the nonlinear dissipative magnetohydrodynamic (MHD) equations with a high-order finite (spectral) element approach \cite{2}. Although particular emphasis is placed on dissipative MHD simulation, the methods developed are generally applicable to hyperbolic and mixed hyperbolic-parabolic systems. In fluid dynamics, open BC are necessary for problems such as pipe flow \cite{3}, propulsion \cite{4}, and weather modeling \cite{5}. Some relevant dissipative MHD problems are plasma propulsion \cite{6,7}, solar coronal physics \cite{8}, and some magnetic confine-ment schemes \cite{9,10}. A discussion of pertinent background information about previous open BC research is presented in Section 2.

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The dissipative MHD equations are presented in Section 3. Three open BC methods for dissipative MHD are implemented and tested in 2D HiFi [11,12], a spectral element code previously known as SEL. The three open BC are approximate Riemann BC (ARBC), lacuna-based open BC (LOBC), and zero normal derivative BC (ZND BC). ARBC are directly based on characteristic analysis of the hyperbolic part of a mixed hyperbolic-parabolic equation system. Building on the work of Ryabenkii et al. [13], LOBC are derived from the concept of surface potentials and Calderon projections. LOBC, in the sense discussed below, control both the hyperbolic and parabolic aspects of mixed hyperbolic-parabolic systems. ZND BC impose zero normal derivative on each variable at the open boundary. Each technique is described in Section 4.

Test problems include pressure pulses in 1D and 2D, a translating field-reversed configuration (FRC) plasma object, and coaxial-electrode plasma acceleration. Comparisons are made to reference cases in which the domain is extended until effects from the distant boundary do not influence the solution in the domain of interest. Test problems and results are described in Section 5. Conclusions are drawn in Section 6.

2. Background

Systematic studies that directly address the problem of open BC for linear or nonlinear dissipative MHD are notably absent. There is, however, a substantial amount of literature describing applications of open BC to hyperbolic systems, including ideal MHD, and some mixed hyperbolic-parabolic systems (e.g., Navier–Stokes). Such literature provides a starting point for work related to dissipative MHD. A paper by Tsynkov [14] provides an extensive review of open BC work. Two review papers by Givoli [15,16] present a thorough overview of the research progress on open BC for linear problems. A review paper on open BC relevant to computational fluid dynamics (CFD) by Colonius [17], briefly covers open BC for linear wave problems and focuses on nonlinear wave problems. Hu [18] also reviews open BC techniques relevant to CFD, and focuses on the perfectly matched layer (PML) technique. The background material presented here is intended to acquaint readers with the ideas and sources from which this research is drawn. For an exhaustive summary of open BC techniques, see the review papers cited above and references therein.

For wave problems where disturbances at the boundary are sufficiently small and smooth, linear open BC are appropriate. Linear open BC are, however, frequently used for nonlinear wave problems [17], sometimes with absorbing layers near the boundary to smooth errors. Thompson [19,20] presents a method based on characteristics that attempts to provide an accurate nonlinear open BC for 2D hyperbolic problems. Colonius [17] points out that Thompson’s BC are not well-posed, but that they work remarkably well in practice.

LeVeque [21], in his book on finite volume methods for hyperbolic problems, describes a technique for specifying non-reflecting BC based on zero-order extrapolation. Information from interior cells is copied to ghost cells located just outside the boundary. Zero-order extrapolation has been successfully employed in past research [22,23], but non-reflection is a consequence of the uniform representation of dependent variables within cells. The technique is not applicable to high-order elements. LeVeque [21] also describes approximate Riemann techniques for treating the characteristics of hyperbolic systems, which form the basis for the approximate Riemann BC (ARBC) technique described in Section 4.1. ARBC are naturally used in finite volume codes that use the approximate Riemann approach at each cell boundary. Discontinuous Galerkin methods, which combine the discontinuous approximate solutions of finite volume methods with finite element representation within cells, also employ ARBC [24]. To the authors’ knowledge, application of ARBC for high-order finite (spectral) element methods, as presented in this research, has not been described in earlier publications.

For hyperbolic systems, well-posedness requires that the incoming waves be constrained at domain boundaries [25]. One BC is required for each incoming characteristic. Because it is variation that constitutes a wave, for non-reflection, incoming wave strength should be constant in time and outgoing waves should be unconstrained. When dissipation is present, the hyperbolic system becomes mixed hyperbolic-parabolic, and BC requirements differ from the purely hyperbolic case [26]. Hesthaven and Gottlieb [27] develop open BC for the Navier–Stokes equations, aiming to ensure well-posedness. They use an energy analysis technique (described in Gustafsson et al. [25]) to develop BC requirements for the continuous problem, and then employ a penalty method to apply the BC in the discrete problem. Nordström and Svärd [28] present a more rate nonlinear open BC for 2D hyperbolic problems. Colonius [17] points out that Thompson’s BC are not well-posed, but that control both the hyperbolic and parabolic aspects of mixed hyperbolic-parabolic systems. ZND BC impose zero normal derivative on each variable at the open boundary. Each technique is described in Section 4.

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There are publications that address open BC for MHD. Dedner et al. [32] discuss an open BC for linear ideal MHD. Their non-local open BC is designed specifically for a gravitationally stratified plasma atmosphere. Faganello et al. [33] use a non-linear open BC following Thompson [19] to minimize boundary effects while studying the effects of magnetic reconnection on the Kelvin–Helmholtz instability. While dissipation and even Hall physics is included in their MHD simulations, they assume that these non-ideal effects are negligible at the simulation boundaries. Forbes and Priest [34] discuss the difficulties of properly specifying open BC in the context of numerical models of magnetic reconnection. None of this work provides clear direction or general prescriptions for setting open BC when the boundary physics is nonlinear and dissipative.

3. Dissipative MHD equations

Open BC methods for dissipative MHD are the focus of this research. The equations of dissipative MHD can be expressed in normalized form as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}
\]

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left\{ \rho \mathbf{v} \mathbf{v} + \left[ \left( \frac{p + B^2}{2} \right) - \mathbf{B} \nabla \cdot \mathbf{v} - \frac{\mathbf{B}}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right] \right] \right\} = 0, \tag{2}
\]

\[
\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \eta \mathbf{j}, \tag{3}
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \left\{ \mathbf{v} \left( e + p + \frac{B^2}{2} \right) - \mathbf{B} (\mathbf{v} \cdot \mathbf{B}) - [\mathbb{A} \eta \mathbf{j} + \mathbb{K} (1 - \mathbf{j} \mathbf{j})] : \nabla \mathbf{v} + \eta \mathbf{j} \times \mathbf{B} \right\} = 0, \tag{4}
\]

where the normalized dependent variables are density ($\rho$), momentum ($\rho \mathbf{v}$), magnetic vector potential ($\mathbf{A}$), and total energy ($e$). $I$ represents the identity tensor. The magnetic field is defined by $\mathbf{B} = \nabla \times \mathbf{A}$, and current density by $\mathbf{j} = \nabla \times \mathbf{B}$. $\zeta$, $\kappa$, and $\kappa_\perp$ are the normalized viscosity, resistivity, parallel thermal conduction, and perpendicular thermal conduction, respectively. These normalized dissipation coefficients are equivalent to inverse Reynolds, magnetic Reynolds, parallel Péclet, and perpendicular Péclet numbers. Thermal conductivities are applied parallel or perpendicular to the magnetic field direction defined by $\mathbf{b} = \mathbf{B} / |\mathbf{B}|$. $\kappa_\perp$ depends on $\mathbf{B}$ and the “magnetized” value of the perpendicular thermal conductivity, $\kappa_\perp$, as follows: $\kappa_\perp = \kappa_\parallel \kappa / (\mathbf{B}^2 \kappa_\parallel + \kappa_\perp)$. As $|\mathbf{B}| \to 0$, $\kappa_\parallel \to \kappa_\perp$. Assuming $\kappa_\parallel \gg \kappa_\perp$, where $\mathbf{B} \approx 1$, $\kappa_\perp \approx \kappa_\perp$. The definition of total energy is $e = p/\gamma + \rho \mathbf{v}^2/2 + \mathbf{B}^2/2$, where $\gamma$ is the ratio of specific heats. $\gamma = 5/3$ is used in this work. The pressure, $p$, is defined in terms of $e$.

4. Open BC formulations

Here, each open BC formulation is presented generically for application to any set of interior partial differential equations (PDEs) in conservation or flux-source form. In places, however, specific reference is made to dissipative MHD (see Section 3).

4.1. Approximate Riemann BC

Approximate Riemann BC (ARBC) are a type of characteristic-based BC (CBC). CBC offer a mathematical basis for specifying boundary conditions. A hyperbolic system is composed of a family of waves, governed by an eigensystem with real eigenvalues corresponding to the wave speeds. CBC identify and appropriately treat the different waves. Hyperbolic systems can be written in a way that allows manipulation of the eigensystem:

\[
\frac{\partial \mathbf{u}}{\partial t} + \sum \mathbb{A} \frac{\partial}{\partial n} \mathbf{u} + \sum \mathbb{A}_t \frac{\partial}{\partial t} \mathbf{u} = 0. \tag{5}
\]

A 2D system is considered, and $\hat{n}$ and $\hat{t}$ refer to the normal and (in-plane) tangential directions at the boundary. The matrices $\mathbb{A}_i$ and $\mathbb{A}_t$ are the flux Jacobians, $\mathbb{A}_n = \frac{\partial \mathbf{F}_n}{\partial \mathbf{u}}$ and $\mathbb{A}_t = \frac{\partial \mathbf{F}_t}{\partial \mathbf{u}}$, and $\mathbf{F}_n$ where $\mathbf{F}_n$ are the normal and tangential flux vectors, respectively. The normal flux Jacobian contains an eigensystem which can be exploited to control the incoming and outgoing waves. $\mathbb{A}_n = \mathbb{R} \mathbb{D}_1$, where $\mathbb{R}$ and $\mathbb{L}$ are matrices of right and left eigenvectors, respectively, and $\mathbb{D}$ is a diagonal matrix of eigenvalues.

Eigensystem decomposition is a challenge, especially for MHD. For this research, both analytical approaches (see Powell et al. [35]) and numerical approaches have been used for decomposing flux Jacobians. The numerical approach is easier to implement – a variety of suitable numerical solver libraries such as PETSc [36] and LAPACK [37] are available.

An upwinding technique from finite volume work (see LeVeque [21]) is used for ARBC. The approach involves “approximately” solving a Riemann problem at the open boundary to determine the normal flux vector. The importance of specifying the normal flux is clarified by considering the integration of a hyperbolic system, written in conservation form, over the entire domain volume.
As a boundary condition for the tangential components of which reduces to an equation describing the evolution of the tangential components of

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F} = 0. \]  

(6)

Using the divergence theorem,

\[ \int_V \frac{\partial \mathbf{u}}{\partial t} dV + \int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = 0. \]  

(7)

This shows that boundary conditions must specify the normal flux at the boundary, \( \mathbf{F}_{\text{bnd}} \equiv \mathbf{F} \cdot \hat{n} \). By convention, the normal unit vector, \( \hat{n} \), points out of the domain.

In ARBC, fluxes corresponding to outgoing waves are determined by the interior solution values. Fluxes corresponding to incoming waves are determined by user-specified ambient conditions just outside of the open boundary. The spectral element code used for this research, HiFi, solves PDEs in the weak form, and a surface term like the one shown in Eq. (7) is explicitly computed for each element.

Roe’s Method [38] is employed to specify the flux in ARBC. In this technique, the normal flux vector at the open boundary is defined as

\[ \mathbf{F}_{\text{bnd}} = \frac{1}{2} (\mathbf{F}_{\text{int}} + \mathbf{F}_{\text{ext}}) - \frac{1}{2} |\hat{\mathbf{A}}| (\mathbf{u}_{\text{ext}} - \mathbf{u}_{\text{int}}), \]  

(8)

where \( \hat{\mathbf{A}} \) is the approximate flux Jacobian (for the normal direction), and \( |\hat{\mathbf{A}}| \equiv \hat{\mathbf{A}} \mathbf{D} \hat{\mathbf{E}} \), where \( |\hat{\mathbf{D}}| \) is a diagonal matrix of absolute values of the eigenvalues. \( \mathbf{F}_{\text{int}} \) and \( \mathbf{F}_{\text{ext}} \) are the fluxes computed using internal and external values, \( \mathbf{u}_{\text{int}} \) and \( \mathbf{u}_{\text{ext}} \). Just as the flux Jacobian, \( \hat{\mathbf{A}} \), is calculated as a function of some average of the variables, \( \mathbf{u} \), \( \hat{\mathbf{A}} \) is calculated as a function of some average of the variables. If \( \hat{\mathbf{u}} \) is chosen well, \( \hat{\mathbf{A}}_{\text{ext}} \mathbf{u}_{\text{ext}} \) and \( \hat{\mathbf{A}}_{\text{int}} \mathbf{u}_{\text{int}} \) are approximately, if not exactly, equal to \( \mathbf{F}_{\text{ext}} \) and \( \mathbf{F}_{\text{int}} \), respectively. Commonly, a simple average of interior and exterior variables is used to find \( \hat{\mathbf{u}} \) (i.e., \( \hat{\mathbf{u}} = \frac{\mathbf{u}_{\text{ext}} + \mathbf{u}_{\text{int}}}{2} \)), although more sophisticated approaches have been studied. For MHD, a simple average is arguably the best option [23].

ARBC allow incoming wave strengths to be determined by exterior conditions (\( \mathbf{u}_{\text{ext}} \)), which are user-specified, and outgoing wave strengths to be determined by interior conditions (\( \mathbf{u}_{\text{int}} \)), which are given by the interior solution values. For instance, if all of the characteristics are outgoing, \( |\hat{\mathbf{A}}| = \hat{\mathbf{A}}_{\text{ext}} \), and \( \mathbf{F}_{\text{bnd}} \) is simply

\[ \mathbf{F}_{\text{bnd}} = \frac{1}{2} (\mathbf{F}_{\text{int}} + \mathbf{F}_{\text{ext}}) - \frac{1}{2} \hat{\mathbf{A}}_{\text{ext}} (\mathbf{u}_{\text{ext}} - \mathbf{u}_{\text{int}}). \]  

(9)

which, after using the approximations \( \hat{\mathbf{A}}_{\text{int}} \mathbf{u}_{\text{int}} \approx \mathbf{F}_{\text{int}} \) and \( \hat{\mathbf{A}}_{\text{ext}} \mathbf{u}_{\text{ext}} \approx \mathbf{F}_{\text{ext}} \), reduces to

\[ \mathbf{F}_{\text{bnd}} \approx \mathbf{F}_{\text{int}}. \]  

(10)

Similarly, if all characteristics are incoming, \( \mathbf{F}_{\text{bnd}} \approx \mathbf{F}_{\text{ext}} \). Effectively, \( \mathbf{F}_{\text{bnd}} = \mathbf{F}^+ + \mathbf{F}_{\text{ext}} \), where the superscript “+” denotes outgoing waves and the superscript “-” denotes incoming waves.

Because ARBC are applied by specifying the flux, the equation system must be in conservation form such that no source terms are present. In the dissipative MHD equations presented in Section 3, total energy is the evolved variable in the energy equation, Eq. (4). The equation is conservative. If Eq. (4) is replaced, for example, with a pressure evolution equation, source terms would be present. Thus, the equation for total energy is preferred. An exception to this rule is made for magnetic field (\( \mathbf{B} \)), which is the conserved variable. Formulation with magnetic vector potential (\( \mathbf{A} \)), where \( \mathbf{B} = \nabla \times \mathbf{A} \), is often preferred to ensure \( \nabla \cdot \mathbf{B} = 0 \). Consider the evolution of \( \mathbf{A} \) and \( \mathbf{B} \) in ideal MHD,

\[ \frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} \]  

(11)

and

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) = 0. \]  

(12)

The quantity to be specified using ARBC is \( \mathbf{F}_{\text{bnd}} = \hat{n} \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) \). Crossing \( \hat{n} \) into Eq. (11),

\[ \hat{n} \times \frac{\partial \mathbf{A}}{\partial t} = \hat{n} \times (\mathbf{v} \times \mathbf{B}), \]  

(13)

which reduces to an equation describing the evolution of the tangential components of \( \mathbf{A} (\mathbf{A}_{\text{tang}}) \) in terms of the boundary flux,

\[ \frac{\partial \mathbf{A}_{\text{tang}}}{\partial t} = -\hat{n} \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) = -\mathbf{F}_{\text{bnd}}. \]  

(14)

As a boundary condition for the tangential components of \( \mathbf{A} \), the time rate of change is specified according to Eq. (14).
4.2. Lacuna-based open BC

Ryabenkii et al. [13] present an open BC approach that takes advantage of lacunae in the solution of hyperbolic problems – lacuna-based open BC (LOBC). The term “lacuna” here refers to a still region behind an aft wave front. Instead of imposing some condition at the boundary, the idea of LOBC is to generate an auxiliary solution that is allowed to propagate into an exterior region appended to the interior domain. The open BC is achieved by constraining the interior solution to match some condition at the boundary, the idea of LOBC is to generate an auxiliary solution that is allowed to propagate into an exterior region.

Fig. 1 shows a schematic for LOBC. Consider a wave problem initialized with non-zero values well inside the domain and zero values near the interior-exterior interface. The auxiliary solution, which is defined only in the exterior and a near-boundary “transition region”, together called the auxiliary domain, is also set to zero initially. Source terms, which drive the auxiliary solution, are generated in the near-boundary transition region such that \( \mathbf{w} = \mu \mathbf{q} \), where \( \mathbf{w} \) is the auxiliary solution, \( \mathbf{q} \) is the interior solution, and \( \mu \) is the transition function.

For an arbitrary set of PDEs in flux-source form, consider the interior problem,

\[
\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S} + \Omega \tag{15}
\]

where \( \mathbf{F} \) is the flux, and \( \mathbf{S} \) is the source. The auxiliary problem is

\[
0 + \nabla \cdot \mathbf{F} = \mathbf{S} + \Omega \tag{16}
\]

Note that the flux and source terms, \( \mathbf{F} \) and \( \mathbf{S} \), can be functions of either \( \mathbf{q} \) or \( \mathbf{w} \). \( \Omega \) is the near-boundary source term that drives the auxiliary problem, and is a function of \( \mathbf{q} \). To determine \( \Omega \), the substitution \( \mathbf{w} = \mu \mathbf{q} \) is made in Eq. (16), and the equation is solved for \( \Omega \). Noting that \( \mu \) is constant in time such that \( \frac{\partial \mu}{\partial t} = 0 \), and using Eq. (15),

\[
\Omega = \nabla \cdot (\mu \mathbf{q}) - \mu \nabla \cdot \mathbf{F} + \mu \mathbf{S} \tag{17}
\]

The following scenario helps illustrate the basic concept. Simulation begins at \( t = 0 \) and the first time step, \( n = 1 \), generates source term \( \Omega_1 \). The problem is stepped forward (or “integrated”) in time, and at each time step, a corresponding \( \Omega_n \) is generated. The wavelet generated by \( \Omega_1 \) reaches the exterior boundary at \( t = T_{ext} \) and \( n = N_{ext} \). Assuming, for the moment, a single wave speed, the time required for a wavelet to travel from the interface to the boundary of the exterior domain is \( T_{ext} = L_{ext}/c \), where \( L_{ext} \) is the length of the exterior domain and \( c \) is the wave speed. The time steps taken are \( n = 1, 2, \ldots, N_{ext} \), corresponding to times \( t = t_1, t_2, \ldots, t_{N_{ext}} \). To prevent the wavelet generated by \( \Omega_1 \) from interacting with the exterior boundary (where a conventional BC is applied), the auxiliary problem is reintegrated, excluding the source term at \( t_1 \). Thus, \( \mathbf{w}_{n=N_{ext}} \) is computed as

\[
\mathbf{w}_{n=N_{ext}} = \mathbf{w}_{n=0} + \int_{t_1}^{t_{N_{ext}}} [\nabla \cdot \mathbf{F} + \mathbf{S} + \Omega] dt \tag{18}
\]

When numerically computing the time integral in Eq. (18), the auxiliary source terms, \( \Omega_n \), computed during the initial integration, may be used in place of \( \Omega(\mathbf{q}) \) at each discrete step. The truncation of the auxiliary source term contribution damps the wave before it reaches the exterior boundary. (Theoretically, it is possible to wait until \( t = 2T_{ext} \), at which point the reflections from the exterior boundary will actually return and influence the interior, but in this research, the earlier reintegration is used.) At the time of truncation, the interior domain is within the lacuna of the truncated wavelet, so the interior solution is not affected (at least in theory). After reintegration, the step \( n = N_{ext} + 1 \) would be taken as usual. All reflections could be prevented by repeating this procedure of reintegration, then stepping. The next reintegration, for example, would be from

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**Fig. 1.** Schematic for lacuna-based open BC (LOBC). In the transition region, the transition function \( \mu \) increases from 0 to 1. The auxiliary domain contains the transition region and the exterior domain.
When \( n = 3 \) to \( n = N_{\text{ext}} + 1 \). However, it is more efficient to take multiple steps between reintegrations. For example, instead of reintegrating from \( n = 2 \) to \( N_{\text{ext}} \), consider reintegrating from \( (\text{the nearest integer to}) \ N_{\text{ext}}/3 \) to \( N_{\text{ext}} \). If the wavelets corresponding to eliminated source terms have exited the transition region by the final time step of the reintegration, \( w \) will match \( q \) at the interior-exterior interface.

An illustration of the time-stepping procedure implemented in this research is given in Fig. 2, where \( N_{\text{ext}} = 9 \) and three steps are taken between reintegrations. Note that the asterisks indicate the steps at which the interior solution, \( u \), is stored. To begin each reintegration indicated by dashed lines, the auxiliary solution, \( w \), is reset to the initial state. During the reintegration per Eq. (18), the solution is re-evolved and used to compute the source term \( \Omega(q) \). An alternative approach would be to store the term \( \Omega(q) \) itself at each discrete time and use the stored values during reintegration. This alternative has the advantage that the solution \( q \) need not be re-evolved.

A slight complication can arise in practice if the choice \( w = \mu q \) is made to compute \( \Omega \) as in Eq. (17). If \( q \) has nonzero initial values in the auxiliary domain, the corresponding initial values for \( w \) will have steep gradients in the transition region, potentially introducing noise immediately in the simulation. This noise can be avoided by choosing \( w = \mu q + q_0 \), where \( q_0 \) is the variation of the solution from background values. The associated source terms drive the auxiliary solution to be \( w = \mu q + q_0 \). If there is no variation of \( q \) (i.e., \( q = 0 \)) initially in the transition region, the auxiliary solution will have a uniform initial state.

In the ideal scenario just described, LOBC are theoretically reflectionless. In practice, however, lacuna-based truncation is often imperfect due to several subtleties:

- **Subtlety #1**: True lacunae exist only when the physics under consideration is odd-dimensional. (This fact is related to Huygens’ principle; see Courant and Hilbert [39].) In 2D, while true lacunae do not exist, they do exist in an approximate sense. A decaying “wake” exists behind 2D waves. As more decay is allowed, lacuna-based truncation becomes more accurate. As decay is allowed, time elapses, and the leading front of the wave travels unimpeded. The exterior domain must be large enough to prevent reflection from the leading front of a wave while the wake is allowed to decay.

- **Subtlety #2**: Lacunae are hyperbolic phenomena and can exist only in purely hyperbolic systems; dissipation modifies lacunae by allowing the waves to diffuse into the lacunae. If the dissipative length scales are comparable to the exterior domain size (\( L_{\text{ext}} \)), error is introduced. If, on the other hand, dissipative scales are much smaller or much larger than \( L_{\text{ext}} \), error will be minimal.

There has been prior work addressing the application lacuna-based open BC when lacunae are imperfect. In an application of lacuna-based artificial BC to electromagnetic waves in dilute plasmas, Tsynkov [40] finds that even when diffusive or dispersive effects cause “weak lacunae,” the lacuna-based BC demonstrates good performance.

- **Subtlety #3**: As discussed in Tsynkov [40] and the references therein, true lacunae exist only for certain wave types – these are sometimes called “Huygens’ waves”. In particular, only equations that are equivalent to the d’Alembert wave equation are Huygens’. For example, in the 1D Euler equations, only the acoustic modes with speeds \( u + c \) and \( u - c \) are Huygens’. The entropy wave with speed \( u \) does not necessarily have lacunae, although it can – consider translation of a compactly supported density fluctuation. Furthermore, the 3D Euler system supports not only acoustic and entropy waves, but also transverse waves (vorticity waves and shear waves), which are not Huygens’.

Presuming for a moment that a hyperbolic system consists of only Huygens’ waves, LOBC can be exact only when the slowest wave is allowed to exit the transition region before truncation. Consider the three wave speeds of the 1D Euler

---

**Fig. 2.** Time stepping procedure for LOBC. The main time integration is shown by solid arrows. Dashed arrows show reintegration. Nine steps are taken before the first reintegration. Three steps are taken between reintegrations. Reintegrations begin with stored solutions indicated by asterisks and results in reintegrated solutions indicated by \( r \) superscripts.
equations: \( u, u + c, \) and \( u - c \). In the case of slow outflow, it may be problematic to wait for all waves to pass through the transition region prior to temporal truncation of the sources – the necessary size of the exterior region could be prohibitively large. The approach taken here is to perform temporal truncation based on the fastest wave speed. By using an exterior region that extends beyond the interior domain by a distance of at least twice the transition length, most slow-moving (or obliquely-moving) wavelets are allowed to pass through the transition region prior to temporal truncation. Despite premature truncation of some wavelets, LOBC perform well if slow-moving (or obliquely-moving) wavelets have small amplitudes such that the truncation leads to small errors.

*Subtlety #4:* In the transition region, the auxiliary solution wave speeds should match those of the interior solution as closely as possible. Mismatched speeds cause disagreement between the auxiliary and interior solutions at the interior-exterior interface, and reflections result. For example, consider the sound speed, which depends on temperature. If the auxiliary pressure is given by \( p^a = \mu \tilde{p} + p_0 \) and auxiliary density by \( \rho^a = \mu \tilde{\rho} + \rho_0 \), the auxiliary temperature is

\[
T^a \approx \frac{\mu \tilde{p} + p_0}{\mu \tilde{\rho} + \rho_0}.
\]

For a linear problem, \( \frac{p_0}{p_0} \approx T^0 \approx T \). In the nonlinear extreme where \( \tilde{p} \gg \rho_0 \) and \( \tilde{\rho} \gg \rho_0 \), again \( T^0 \approx T \). If variations \( (\tilde{p} \text{ and } \tilde{\rho}) \) are near the background values \( (p_0 \text{ and } \rho_0) \), \( T^0 \neq T \). For this case, lower background values can sometimes be used so that the extreme nonlinear case is approached. Similarly, when using an internal energy formulation for gas dynamics or MHD, care should be taken to ensure that \( p^a = \mu \tilde{p} + p_0 \). Auxiliary pressure is defined as

\[
p^a = (\gamma - 1)(e^a - KE^a - ME^a),
\]

where \( e^a \) is the auxiliary total energy, \( e^a = \mu \tilde{e} + e_0 \), \( KE^a \) is the auxiliary kinetic energy, \( KE^a = (m^a)^2/(2 \rho^a) \), where \( m \) is momentum, and \( ME^a \) is the auxiliary magnetic energy, \( ME^a = (B^a)^2/2 \), where \( B \) is magnetic field. Assuming that the background flow speed is zero, \( m^a = \mu \tilde{m} \). As usual, \( \rho^a = \mu \tilde{\rho} + \rho_0 \). If \( B^a = \mu \tilde{B} + B_0 \), auxiliary pressure is

\[
p^a = (\gamma - 1) \left( \mu \tilde{e} + e_0 - \frac{(\mu \tilde{m})^2}{\mu \tilde{p} + p_0} - \frac{(\mu \tilde{B} + B_0)^2}{2} \right).
\]

Here, \( e_0 = p_0/(\gamma - 1) + B_0^2/2 \). In the linear case, Eq. (21) yields \( p^a \approx p_0 \), resulting in \( T^0 \approx T_0 \). In the nonlinear case with background values small relative to variations, \( e_0 \approx \mu e \), and \( KE^e \approx \mu KE \), but \( ME^e = \mu^2 ME \). Thus, \( p^a \neq \mu p \), and \( T^0 \neq T \). This problem is corrected by generating the source term for the magnetic evolution equation with \( \sqrt{\mu} \). Then, \( ME^a = \mu ME \) and the auxiliary solution temperature matches the interior solution.

A goal of this research has been to explore these subtle complications related to lacuna-based truncation and determine whether LOBC can be useful despite them. As shown in Section 5, results indicate that LOBC are indeed useful for dissipative MHD simulation.

Smoothness requirements of the transition function depend on the physical system being modeled. For the Euler equations, for example, continuous first derivatives are needed – if the auxiliary pressure has a discontinuous derivative, the associated discontinuous force \( (\nabla \cdot \mathbf{p}) \) would be problematic. For magnetic-vector-potential-based MHD, continuous second derivatives are required to ensure a smooth representation of auxiliary current. The transition function, \( \mu \), used here is based on a quintic polynomial. For a transition of length \( L_{\text{trans}} \), between points \( x_0 \) and \( x_1 \), that polynomial is

\[
P_5 = \Lambda^4 (10 - 15 \Lambda + 6 \Lambda^2),
\]

where \( \Lambda = (x - x_0)/L_{\text{trans}} \). For LOBC, a function is sought which has \( C^2 \)-continuity and which has a \( C^2 \)-continuous square root. \( P_5 \) is \( C^2 \)-continuous, but \( P_5^{1/2} \) is not. However, \( P_5^{1/2} \) and \( P_5^{2/4} \) are \( C^2 \)-continuous. The transition function \( \mu = P_5^{1/2} \) is used, along with its square root for magnetic variables. To minimize noise in the spatial representation of \( \mu \), the transition region is designed so that its limits correspond to cell boundaries. Generally, two or more cells are used in the transition region.

Direct replacement of \( q \) with \( w \) in the exterior after reintegration could introduce discontinuities if the solutions do not perfectly match. This problem is alleviated by gradual replacement – \( q \) is unmodified at the interior-exterior interface, and smoothly transitions to \( w \) at a distance of one transition length from the interface into the exterior region.

### 4.3. Zero normal derivative BC

Zero normal derivative boundary conditions (ZND BC) enforce zero normal derivative for each dependent variable at the open boundary. As shown in Section 5, ZND BC allow significant reflections, but are stable for dissipative MHD and provide a good point of comparison for more sophisticated open BC.

### 5. Test problems and results

The capabilities and performance of the open BC options presented in Section 4 are explored in three test problems, each of which involves solving dissipative MHD (see Section 3) or the gas dynamics subset of dissipative MHD. Performance of the three open BC is quantified in terms of the proximity of their associated solutions to a reference solution. Reference solutions are obtained by solving the problems on greatly extended domains to prevent boundary effects. The BC used for the
reference cases are described specifically for each problem. To ensure that the reference case BC do not affect the interior solution, reference domain size is increased until variation in the obtained interior solution is negligible. Performance is measured in terms of the \( L_\infty \)-norm of the pressure error (i.e., the maximum error) in the simulation. In plotted results, \( L_\infty \) error is normalized by the maximum pressure during the reference simulation.

The time advance method used in all test problems is the fully implicit Crank–Nicolson scheme (i.e., equations are all advanced simultaneously using the theta method with \( \theta = 1/2 \)).

### 5.1. 1D and 2D pressure pulse propagation

Fig. 3 depicts the setup for the 2D pressure pulse problem. The 1D pressure pulse problem is the restriction of the 2D problem to \( y = 0 \). A constant temperature pressure pulse is initialized at the center of the domain. The fluid is initially at rest. Background pressure and density are \( p_0 = \rho_0 = 1 \). The wave speed used to dictate the LOBC truncation is the sound speed in the background fluid, \( c_s = \sqrt{\gamma p_0 / \rho_0} \), where \( \gamma = 5/3 \) is the ratio of specific heats. A localized perturbation with peak pressure \( p_{max} = p_0 + \delta \) and peak density \( \rho_{max} = \rho_0 + \delta \) is initialized at the center of the domain, as shown in Fig. 3. Small and large perturbations, \( \delta = 10^{-3} \) and \( \delta = 0.5 \), are used to test linear and nonlinear regimes. In both linear and nonlinear cases, the dissipation coefficients are \( \xi = 10^{-3} \), and \( \kappa_\parallel = \kappa_\perp = 10^{-3} \) – relatively small values such that the LOBC should be unaffected by the concerns related to Subtlety #2 of Section 4.2. For the reference case, the following BC are used: \( \mathbf{n} \cdot \mathbf{v} = 0 \) (hard wall); \((\mathbf{n} \cdot \nabla)(\mathbf{n} \times \mathbf{v}) = 0\) (perfect slip); and \( \mathbf{n} \cdot \nabla T = 0 \) (thermally insulating). These BC are also used at the exterior boundary for the LOBC case. The computational grids have eight cells per unit length, each with 4th-degree polynomials. The time step size is fixed as \( dt = 10^{-2} \).

Fig. 4 shows a series of snapshots of the absolute value of pressure \( |p - p_0| \) for the case with ARBC truncation. The nonlinear case (\( \delta = 0.5 \)) is depicted. By showing the absolute value, a logarithmic scale can be used to help reveal small reflections. The pressure pulse expands, leaving a qualitatively uniform solution in its wake. As the pulse passes through the boundaries, no reflection is readily observable. Late in time, small pressure deviations caused by reflections are seen in the domain. Qualitatively, results are similar for TBC and LOBC – small amplitude reflected waves rebound through the domain. For the ZND BC, rebounding waves are more substantial as shown in the quantitative results below. The time advance method used in all test problems is the fully implicit Crank–Nicolson scheme (i.e., equations are all advanced simultaneously using the theta method with \( \theta = 1/2 \)).

Fig. 5 presents 1D results for linear (\( \delta = 10^{-3} \)) and nonlinear (\( \delta = 0.5 \)) cases. In 1D, errors for ARBC and LOBC are higher in the nonlinear case. ARBC are based on linearization of the problem and lose effectiveness for nonlinear waves. As discussed in Section 4.2, the accuracy of LOBC is reduced in the regime between linear (\( \delta \ll 1 \)) and highly nonlinear (\( \delta \gg 1 \)) extremes. This reduced accuracy could explain the higher error for LOBC in the nonlinear case. The linear and nonlinear results in 2D are presented in Fig. 6. The fact that the results are nearly identical implies that the main source of error is not the nonlinearity for any of the open BC in 2D. For LOBC, the higher error seen in the 2D results is presumably because of the lack of true lacunae in 2D (see Subtlety #1 of Section 4.2). For ARBC, higher error is also seen in 2D, possibly because of the oblique nature of the waves interacting with the boundary – ARBC are designed to treat only waves traveling in the normal direction. Oblique (or “transverse”) waves can cause reflections. For ZND BC, higher error is seen in the 1D results than in the 2D results. In two dimensions, the wave strength diminishes as the wave spreads radially, but the strength remains constant in one dimension. In 2D, a weaker wave interacts with the BC, and less reflection occurs.

**Fig. 3.** Computational domain and initialization for 2D pressure pulse propagation. The 1D pressure pulse problem is the restriction of the 2D problem to \( y = 0 \). Symmetry BC are used at the left and bottom of the computational domain. Dashed contours indicate initial pressure. For ARBC and ZND BC tests, only the interior (i.e., the region inside the heavy dashed line) is modeled. For the LOBC, the interior and exterior are modeled. The LOBC transition region is at the edge of the interior region, where \( 0.75 < x < 1 \) and \( 0.75 < y < 1 \). Solid contours indicate the value of the transition function, \( \mu \). Simulations with \( L_{ext} = 0.5 \) and \( 2.0 \) are run.
Fig. 4. Snapshots of the absolute value of the perturbed pressure ($p - p_0$) for the nonlinear pressure pulse problem with ARBC domain truncation. At time $t = 0$, the initial condition is seen in the $1 \times 1$ square domain. At $t = 0.4$ and $t = 0.8$, a pressure wave expands and passes through the open boundary. The ARBC produces no observable reflection until late in time. At $t = 1.6$, small perturbations are observed to persist in the domain.

Fig. 5. 1D pulse problem results for linear ($\delta = 10^{-3}$) and nonlinear ($\delta = 0.5$) cases. $L_\infty$ error, measured with respect to reference cases, is plotted vs. normalized time. For ARBC and LOBC, error increases in the nonlinear case, but remains below 1%. Errors for ZND BC are over an order of magnitude higher than all other open BC options. LOBC with $L_{ext} = 2.0$ performs best for both cases with error less than 0.1%.

Fig. 6. 2D pulse problem results for linear ($\delta = 10^{-3}$) and nonlinear ($\delta = 0.5$) cases. $L_\infty$ error, measured with respect to reference cases, is plotted vs. normalized time. Errors are nearly identical for linear and nonlinear cases, suggesting that the nonlinearity is not the main source of error for any of the open BC in 2D (see discussion in text). ZND BC produce errors an order of magnitude higher than the other open BC. LOBC with $L_{ext} = 2.0$ performs best for both cases with error less than 1%.
The following conclusions can be drawn from the pressure pulse propagation results. ZND BC results show high $L_\infty$ error in 1D and 2D as compared to the other two open BC considered. ARBC performance is similar to LOBC with $L_{\text{ext}} = 0.5$. The LOBC with $L_{\text{ext}} = 2.0$ consistently performs best for 1D, 2D, linear, and nonlinear cases.

5.2. Field-reversed configuration translation

The field-reversed configuration (FRC) [41] is a magnetic plasma confinement configuration analogous to the Hill’s vortex. Where the Hill’s vortex has closed streamlines that confine fluid density, the FRC has closed magnetic field lines that confine hot plasma.

FRC translation is important for various experiments including the electrodeless Lorentz-force (ELF) thruster program [42]. The ELF thruster concept generates thrust by repeated FRC formation, high-speed translation, and ejection. An open BC is appropriate for ELF modeling. To address the challenge of allowing an FRC plasma to exit an open boundary, the simple FRC translation problem depicted in Fig. 7 is used. The initial condition for this problem is generated using a numerical equilibrium solver developed by Marklin [43]. The peak density and pressure of the FRC are

$$\rho_0 = 5 \times 10^{-3} \quad \text{and} \quad p_0 = 0.071.$$  

The magnetic field strength at the ends of the interior domain, where the field has no axial variation, is $B_{\text{ext}} = 0.6$. Thermal conduction is highly anisotropic, as expected physically in magnetized plasma. Values for dissipation coefficients are $\kappa_\parallel = 1, \kappa_\perp = 0.01, \zeta = 0.01,$ and $\eta = 2 \times 10^{-3}$. The wave speed used to dictate the LOBC truncation is the fast MHD wave speed in the background fluid, $C_f$, found by the relation $C_f = \left(\gamma p_0 + B_{\text{ext}}^2/2\right)/p_0$, where $\gamma = 5/3$ is the ratio of specific heats. In terms of the dependent variables and the boundary normal $(\hat{n})$, the radial wall BC are $\hat{n} \cdot \nabla = 0$ (hard wall); $(\hat{n} \cdot \nabla)(\hat{n} \times \mathbf{v}) = 0$ (perfect slip); $\partial(\hat{n} \times \mathbf{A})/\partial t = 0$ (perfectly conducting); and $\mathbf{n} \cdot [\kappa_\parallel \mathbf{b} + \kappa_\perp (1 - \mathbf{b})] \cdot \mathbf{v} = 0$ (thermally insulating). For the LOBC, at the exterior boundaries, ZND BC are applied. The computational grids have two cells per unit axial length, and eight cells per unit radial length, each with 6th-degree polynomials. The time step size is fixed as $dt = 10^{-2}$.

As mentioned in Section 4.2 for LOBC, ideally, the external domain size ($L_{\text{ext}}$) should not match the dissipation scale length. In the pressure pulse problems shown, the dissipative scales are much smaller than $L_{\text{ext}}$ but here, parallel thermal conduction is high. The relevant dissipation scale length can be estimated as $L_{\text{ext}} \approx (T_{\text{ext}}\kappa_\parallel)^{1/2}$, where $T_{\text{ext}}$ is the time that source terms are kept in the LOBC, in this case, $T_{\text{ext}} = L_{\text{ext}}C_f \approx 0.39$. Therefore, $L_{\text{ext}} \approx 0.62$, while $L_{\text{ext}} = 2.0$.

Results for FRC translation are given in Fig. 8. The ARBC run fails after a few time steps; strong thermal conduction parallel to magnetic field lines interacts non-physically with the ARBC and density is driven to zero. The simulation becomes numerically intractable as characteristic speeds approach infinity. The ZND BC run is numerically stable, and the maximum $L_\infty$ error is 3%. As for the pressure pulse results shown in Fig. 4, errors introduced by the LOBC and ZND BC rebound from the boundary and affect the entire interior region. In this problem, strong parallel thermal conduction rapidly diffuses the errors through the interior domain. Although the $L_k \ll L_{\text{ext}}$ condition is not satisfied, the maximum LOBC error is only 1%. While strong dissipation can cause reflection error – see Subtlety #2 discussed in Section 4.2 – perhaps the dissipation significantly smooths that error. Note that the results of this problem provide evidence related to Subtlety #3 of Section 4.2. That is, many wave speeds are present as the FRC translates through the open boundary, but truncation based on the fastest speed provides an accurate solution. Qualitatively, it is observed that the highest-pressure portion of the FRC interacts with the open BC at $t \approx 5$.

5.3. Coaxial-electrode plasma acceleration

Coaxial-electrode plasma acceleration is a process common to a variety of plasma experiments and applications including plasma formation in the ZaP flow Z-pinch experiment [9], magnetoplasmadynamic (MPD) thrusters [44,45], and plasma gun
spheromak formation \cite{46, 47} to name a few. The plasma acceleration problem setup is shown in Fig. 9. The geometry and plasma acceleration parameters are similar to those present in the ZaP acceleration region. The background pressure and density are \( p_0 = 10^{-3} \) and \( \rho_0 = 0.1 \). A density concentration with peak density \( \rho = 5 \) is centered at a distance of 0.2 from the left boundary. Dissipation coefficients are \( \kappa = \kappa_i = 0.04 \) (isotropic thermal conduction), \( \eta = 2 \times 10^{-3} \), and \( \zeta = 0.04 \). Flux injection is achieved by specifying the azimuthal magnetic field on the left boundary with an inverse radial dependence, \( B_0(r,t) = B_0(t) r^{-\frac{1}{2}} \), where \( r \) is the radius of the inner electrode. In this way, the total current driven between the electrodes is specified as \( I = 2 \pi a B_0(t) \). Momentum at the left boundary is held to zero. A resistive layer is present at the left boundary, with \( \eta \) rising from \( 2 \times 10^{-3} \) to 0.1 over a distance of 0.1. The functional form of the rise is a half-period (trough to crest) of a sinusoid. The plasma accelerates to a maximum speed of approximately 6.67 times the background sound speed, \( c_s \). Therefore, the speed used to dictate LOBC truncation is 6.67 \( c_s \). At the inner and outer electrodes, the BC are identical to those applied for the FRC problem described in Section 5.2 – hard wall, perfect slip, perfectly conducting, and thermally insulating. In the reference case for this problem, this set of BC is also applied at the right boundary. At the exterior boundary of the LOBC, ZND BC are applied. The computational grids have 64 cells per unit axial length (i.e., 40 cells in the interior domain, which has a length of 0.625), and four cells across the radial extent of 0.05. Each cell has 6th-degree polynomials. The time step size is fixed as \( dt = 5 \times 10^{-4} \).

A cyclic simulation is performed. Current is cycled up and down, and a cyclic density source, approximating gas injection and ionization, is used to replenish the plasma. The density source has the same spatial profile as the original slug, and the total density added in each cycle is equal to the original density. The current and density source profiles are shown in Fig. 10. The maximum enclosed current is \( I_{\text{max}} = 1.9 \). In this scenario, without an open BC, after many cycles, a prohibitively large computational domain would be required to prevent the influence of reflections from the downstream boundary on the solution in the “interior” region shown in Fig. 9.

The \( L_{\infty} \) results during four plasma acceleration cycles (per Fig. 10) are shown in Fig. 11. As in the FRC translation problem, in the presence of significant dissipation, the ARBC fails. Again, LOBC outperform ZND BC. In the first acceleration event, which has peak kinetic energy at \( t \approx 0.4 \), error for the LOBC is 0.7% and 6% for the ZND BC. For later acceleration events at \( t \approx 1.4, 2.4, \) and 3.4, reflection is higher for LOBC and ZND BC. This can be explained by the fact that current sheet propagation speed is sensitive to the low-density plasma wake left behind the previously accelerated sheet. Errors introduced by the LOBC and ZND BC rebound from the boundary and affect the entire interior region. The relatively high ZND BC error

![Fig. 9. Coaxial plasma acceleration simulation setup. As shown with dashed contours, density is concentrated in a “slug” with peak density \( \rho = 5 \) near the left boundary. A current supply drives flux injection through an insulating boundary at the left end. A current sheet forms, and the plasma slug is heated and driven axially toward the open boundary at the right end. For ARBC and ZND BC tests, only the interior (i.e., the region inside the heavy dashed line) is modeled. For the LOBC, the interior and exterior are modeled. The LOBC transition region is at the edge of the interior region, where \( 0.5 < z < 0.625 \). Solid contours (vertical lines) of \( \mu \) are shown in the transition region.](image-url)
allows the low-density plasma to deviate significantly from the reference case, causing $L_\infty$ error near 100% for the second, third, and fourth acceleration events. The LOBC gives current sheet wake properties closer to the reference case, and error for the second, third, and fourth acceleration events ranges from 2% to 17%. As for the FRC translation problem reported in Section 5.2, the results for this problem provide evidence related to Subtlety #3 of Section 4.2 – despite the presence of a variety of wave speeds, LOBC truncation based on the fastest speed provides an accurate solution.

6. Conclusions

Three methods for modeling open BC have been described. The first method, approximate Riemann boundary conditions (ARBC), locally computes fluxes using an approximate Riemann technique to specify incoming wave strengths. In the second method, lacuna-based open boundary conditions (LOBC), an exterior region is attached to the interior domain where hyperbolic effects are damped before reaching the exterior region boundary where the remaining parabolic effects are bounded using conventional BC. The third method, zero normal derivative BC (ZND BC), enforce zero normal derivative on each dependent variable at the open boundary.

Three test problems conducted with a spectral element code, HiFi, demonstrate the open BC. Boundary reflection is quantified for ARBC, LOBC, and ZND BC by comparing associated solutions to a reference solution computed in a domain large enough to prevent undesired boundary effects. 1D and 2D pressure pulse problems test linear and nonlinear regimes. In the pressure pulse problems, ARBC and LOBC outperform ZND BC, giving normalized $L_\infty$-norm pressure errors less than 5%. LOBC performance is found to improve as the exterior region size is increased.

An FRC translation problem with strong thermal conduction, and a coaxial plasma acceleration problem with high-speed, high-gradient flow, are more challenging for the open BC. In both of these problems, LOBC excel where ARBC fail due to the presence of strong dissipation.

ZND BC are by far the easiest to implement of the three open BC. However, for problems that are sensitive to boundary effects, ZND BC could be inadequate.

In the pressure pulse problems of Section 5.1, which are dominantly hyperbolic, ARBC consistently outperform ZND BC. Other advantages and disadvantages of ARBC are:

![Fig. 10. Profiles of the cyclic current and density source rate (normalized by their respective maximum values) for the coaxial plasma acceleration problem. Over each normalized time unit, current is sinusoidally ramped up and down. After the current has swept downstream, the density source replenishes the plasma at the location of original concentration.](image)

![Fig. 11. Results for four cycles of coaxial-electrode plasma acceleration. $L_\infty$ error, measured with respect to a reference case, is plotted vs. normalized time. Dissipation causes numerical instability of ARBC around $t = 0.4$. ZND BC and LOBC are stable with maximum $L_\infty$ errors near 100% and 10%, respectively.](image)
ARBC implementation is more complicated than ZND BC, but significantly less involved than LOBC.

In the FRC translation and coaxial acceleration problems of Sections 5.2 and 5.3, parabolic effects are significant at the open boundaries, and ARBC do not properly bound the system behavior, allowing numerical instability.

Oblique-moving waves are not properly treated in this technique, which is designed for waves moving normal to the open boundary.

When ARBC fail in the presence of dissipation, LOBC provide an open BC option that generates significantly less reflection than ZND BC, as demonstrated in the test problems. Other advantages and disadvantages of LOBC are:

- Implementation of LOBC is complicated and problem-dependent.
- As discussed in detail in Section 4.2, exact non-reflection with LOBC can be lost for a variety of reasons: in 2D problems, true lacunae do not exist; dissipation modifies lacunae; auxiliary solution components associated with slow-moving or obliquely moving waves must often be prematurely truncated; nonlinearity can cause a mismatch of the wave speeds of the auxiliary solution with the interior solution.
- Even when exact non-reflection is not possible, the test problems show that significantly lower reflection error is achievable with LOBC than with either ARBC or ZND BC.
- Because the LOBC algorithm relies on repeated reintegration from a base state, the method is not subject to the long-time instabilities that can be problematic for alternative algorithms such as the perfectly-matched layer (PML). In fact, as mentioned in Section 2, Qasimov and Tsynkov [31] have demonstrated that lacuna-based methods can be used to stabilize PML for electromagnetic problems.

The challenges of implementing LOBC as an open BC for dissipative MHD have been explored. While known alternatives either produce undesirable reflections or are numerically unstable, LOBC have proven to be effective.

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