Cooperative Target Tracking using Oscillator Models in Three Dimensions

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Abstract—The work in this paper addresses the task of control design for coordinated autonomous vehicles operating in 3D. A group of \( N \) unit-speed individuals are modeled using Natural Frames, in which each individual has a total of two inputs that act to steer the vehicle gyroscopically. Extending previous results for planar target tracking based on oscillator models, a velocity matching controller is first designed for each individual so that the collective centroid is asymptotically stabilized to a moving target vehicle. Simultaneously, an additive spacing control term induces helical motion in order to keep pursuing vehicles near the centroid. Simulation examples are presented to support analytical results.

I. INTRODUCTION

The ability to track moving targets is one of the fundamental applications of multivehicle control theory. Although the individual vehicles need not coordinate their motions at all, doing so can improve the performance of the group. Multiple observers of a target object provide robustness against vehicle loss in addition to providing improved estimates of the target object state. In the present work, coupling between individual vehicles is achieved through control of their collective centroid. Collocation of the centroid and a target ensures that vehicles are distributed around the target as opposed to being clustered on one side, as can happen without coordination.

The individual vehicles considered here are both nonholonomic and constant-speed. The target object, however, need not be constant-speed nor nonholonomic and typically has a speed less than that of the \( N \) pursuer vehicles. The objective of this work is to drive the group centroid to the target, the centroid velocity to the target velocity, and to simultaneously keep individuals near the target.

Three-dimensional target tracking has been considered only recently in the control literature. Justh and Krishnaprasad [1] considered the related problem of controlling two or more vehicles to achieve circular, helical, and rectilinear formations. A key contribution of their work was the use of Natural Frenet Frames to represent vehicle dynamics. The foundation for the controls developed here utilizes this Natural Frame modeling approach in addition to steering control based on the ideas of the Kuramoto model of non-linearly coupled oscillators [2]. Coupled oscillator models were introduced in the physics community over forty years ago to study synchronization of large distributed systems [3].

Applied to planar vehicle models by treating heading angle as oscillator phase, the Kuramoto model provides simple controls that, for example, can align vehicle headings or can drive the centroid velocity to zero (balance). This approach has been used in several instances to develop formation controls for planar vehicles [4], [5], [6], [7], [8].

A technique of complementary velocity matching and formation controls was developed by Klein and Morgansen [9] for three planar vehicles. In that work, straightforward group velocity matching controls drive vehicles away from targets moving at less than pursuer speed, so target tracking requires the development of formation controls to keep pursuers near the target without affecting the group centroid. The extension of dynamic velocity matching control to vehicles in three dimensions is straightforward, and this is the basis for the approach to target tracking taken here.

The contributions of this paper are three-fold. First, an extension of Kuramoto-based control laws to achieve aligned and balanced conditions of the headings of a group of vehicles in three dimensions is made. To achieve this task, vehicle dynamics are represented with Natural Frenet Frames which provide an appropriate coordinate system for simple gradient-descent control laws. The second contribution is the development of a dynamic velocity matching control that does not interfere with formation control. The final contribution is the development of helical formation controls applicable to the task of maintaining proximity to a target without interfering with velocity matching control (Fig. 1).

This paper is organized as follows. An overview of the problem formulation, including a description of the vehicle dynamics, assumptions, and control design is given in Section II. In Section III, the classical planar coupled oscillator model is extended to three dimensions. Based

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Fig. 1. An example of three vehicles tracking a target using steering controls based on oscillator models. Helical motion is induced to keep the pursuing vehicles near the group centroid. The centroid has stabilized to the target.
on this extension, a controller that asymptotically stabilizes the centroid velocity to a reference velocity is developed in Section IV. The reference velocity is in turn set by an outer loop controller developed in Section V. In Section VI, an additive control term that results in helical motion is proposed to keep individuals near the target. Final remarks and future work are discussed in Section VII.

II. PROBLEM FORMULATION

A. Dynamical System Model

Consider a homogeneous group of $N$ constant-speed nonholonomic vehicles moving in three dimensions. Using Natural Frenet Frame dynamics, each of these vehicles is modeled as a point tracing out a curve in Cartesian space. In this model, the position of a vehicle is represented by a vector, $r$, and the orientation is represented by an orthonormal frame formed of a unit tangent vector, $t$, a unit normal vector, $n$, and a unit binormal vector, $b$. After a scaling of time, the constant-speed case simplifies to unit-speed in which the velocity of the vehicle is, at any instant, equal to the tangent vector. More formally, the Natural Frame model can be written as

$$\frac{d}{dt} \begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & u \\ 0 & -u & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \quad (1)$$

where $u$ and $v$ are control inputs that cause the vehicle to steer towards the normal and binormal vectors, respectively. The dynamics (1) are often used in the literature to represent nonholonomic, unit-speed vehicles [1], [9], [10]. The term system will be used to refer to $N$ such vehicles with dynamics as in (1) and a single target vehicle whose position, velocity, and acceleration are denoted $\dot{r}_t$, $\ddot{r}_t$, and $\dddot{r}_t$. References to group states, such as the position and velocity of the centroid, include the $N$ pursuer vehicles but not the target. The control laws designed in the following sections are based on a reference velocity vector, $\dot{r}_{ref}$, which is not to be confused with the velocity of the target vehicle.

B. Assumptions

Several assumptions are needed in order to allow for tractable analysis. The path of the target vehicle is assumed to be at least twice continuously differentiable. Each pursuing vehicle is assumed to have full information about the target vehicle including its position, velocity, and acceleration. Further, every pursuer is assumed to have continuous, undelayed access to the state of every other pursuer. This assumption is a reasonable approximation of the information that could be obtained through all-to-all communication, all-to-all sensing, and/or a local state estimator. Finally, without loss of generality, the speed of each pursuer vehicle is set to one, and the speed of the target vehicle is restricted to $|\dot{r}_{t}| \in [0, 1)$. These last specifications ensure that the pursuers can always catch up with the target.

C. Control Design Overview

The control inputs $u_k$ and $v_k$ on each vehicle, $k \in \{1, \ldots, N\}$, are developed incrementally by considering subproblems of increasing difficulty that are combined to solve the overall problem. These subproblems are as follows:

1) Constant Velocity Matching: Steer each vehicle such that the velocity of the group centroid matches a known constant reference velocity, $\dot{r}_{ref}$. $\ddot{r}_{ref}$. The acceleration of the reference velocity is assumed known.

2) Dynamic Velocity Matching: Steer each vehicle such that the centroid velocity matches a known dynamic reference velocity, $\dot{r}_{ref}$. The acceleration of the reference velocity is assumed known.

3) Centroid Target Tracking: Define an outer loop controller to generate an appropriate reference velocity to stabilize the position and velocity of the group centroid to the position and velocity of the target vehicle.

4) Target Tracking: Apply a spacing controller to keep each vehicle near the collective centroid without interfering with velocity matching controls.

The resulting composite controller achieves target tracking in that the group centroid tracks the target vehicle, while at the same time, the pursuing vehicles stay near the target.

III. COUPLED OSCILLATOR MODELS FOR STEERING CONTROL

Phase coupled oscillator models were first studied by Winfree [11] and later by Kuramoto [2] and Strogatz [3], [12], [13], [14] as a tool for analyzing large groups of coupled systems. In the most generic form with all-to-all coupling, each oscillator has a single state variable $\theta$, that evolves according to

$$\dot{\theta}_k = \omega_k - K \sum_{j=1}^{N} \sin(\theta_j - \theta_k). \quad (2)$$

Here, $K$ is a gain, $\theta_k$ is the phase of the $k^{th}$ oscillator, and $\omega_k$ is its fixed natural frequency. In the case of homogeneous natural frequencies ($\omega_k = \omega_0 \forall k$), $\omega_0$ can be taken as zero without loss of generality by rewriting (2) in a rotating frame. A useful term in the study of coupled oscillators is the order parameter [2],

$$R(\theta) = \frac{1}{N} \left\| \sum_{j=1}^{N} \left[ \cos \theta_j \right] \right\|. \quad (3)$$

This parameter quantifies the amount of order in the system, taking the maximal value of one when all the oscillators are synchronized, termed the aligned set, and the minimal value of zero in the balanced set. In the case of a homogeneous natural frequency of value zero, the phase coupled oscillator model (2) is closely related to the order potential,

$$U(\theta) = \frac{N}{2} R(\theta)^2. \quad (4)$$

In fact, for the system $\dot{\theta}_k = u_k$, the coupling model (2) can be realized using a scaled gradient controller based on the
order potential:
\[
\dot{\theta}_k = u_k = -K \frac{\partial U}{\partial \theta_k} = -\frac{K}{N} \sum_j \sin(\theta_j - \theta_k).
\] (5)

The state of the $N$-oscillator system is driven to the balanced set for $K > 0$ and to the aligned set for $K < 0$.

Focusing once again on vehicles, the state of each individual must describe its position and orientation. Restricting the three-dimensional dynamics (1) to the planar case by choosing $b = 0$ gives
\[
\begin{bmatrix}
\dot{r} \\ \dot{t}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\ 0 & 0 & u \\ 0 & -u & 0
\end{bmatrix}
\begin{bmatrix}
\dot{r} \\ \dot{t}
\end{bmatrix},
\] (6)
or equivalently
\[
\frac{d}{dt}
\begin{bmatrix}
r_x \\ r_y \\ \theta
\end{bmatrix} =
\begin{bmatrix}
\cos \theta \\ \sin \theta \\ u
\end{bmatrix}.
\] (7)

To clarify the connection between oscillator models and planar vehicle dynamics, consider again the order potential (4). The unit tangent vector for a planar vehicle $k$ is defined as $\vec{t}_k \equiv [\cos \theta_k \ \sin \theta_k]^T$, so that the order potential can be rewritten as
\[
U(\vec{g}) = \frac{N}{2} \|\vec{t}(\vec{g})\|^2
\] (8)
where $\vec{g} \in SO(2)^N$ is the orientation portion of the state of the group, and
\[
\vec{t}(\vec{g}) = \frac{1}{N} \sum_{k=1}^N \vec{t}_k
\] (9)

is the centroid velocity. For notational brevity, the explicit dependence of $\vec{t}$ on $\vec{g}$ will be suppressed. The maximal value of $U$ corresponds to all vehicles being aligned whereas the minimal value corresponds to a balanced state where the centroid has zero velocity. Note that the order potential is not a function of the vehicles’ positions and that $\frac{\partial U}{\partial r_k} = \vec{t}$. For the planar scenario, a system which subsumes (2) with homogeneous natural frequency can be derived by choosing the heading control of each vehicle to follow the negative gradient of the order potential:
\[
\dot{t}_k = u_k \vec{n}_k = -K \frac{\partial U}{\partial t_k} = -K \vec{t}.
\] (10)
The scalar controls, $u_k$, can be isolated by pre-multiplying both sides of (10) by $\vec{n}_k^T$, resulting in
\[
u_k = -K \vec{n}_k^T \vec{t}.
\] (11)

When the inner product is expanded using the definition of $\vec{t}$, the resulting control is equivalent to (5).

The analogous development of systems in 3D is straightforward. The definition of centroid velocity and order potential remain the same, although they gain one dimension. Again setting the velocity of each vehicle to the scaled gradient of the order potential,
\[
\dot{t}_k = u_k \vec{n}_k + v_k \vec{b}_k = -K \frac{\partial U}{\partial t_k} = -K \vec{t}.
\] (12)

Because $\vec{n}_k$ and $\vec{b}_k$ are orthonormal, $u_k$ and $v_k$ can be isolated by pre-multiplying by $\vec{n}_k^T$ and $\vec{b}_k^T$, respectively, to obtain
\[
\begin{align*}
u_k &= -K \vec{n}_k^T \vec{t} \\
v_k &= -K \vec{b}_k^T \vec{t}.
\end{align*}
\] (13)

This control will drive all vehicles to a balanced or aligned state, depending on the sign of $K$.

IV. MATCHING A REFERENCE VELOCITY

The task of asymptotically stabilizing the velocity of the group centroid to a reference velocity is considered in this section. First, the reference velocity is assumed to be constant, and a controller based on oscillator models is developed. Then, an additive feed-forward control term is designed to permit asymptotic stability in the case of a dynamic reference velocity. Note that no target vehicle position is defined at this point, only a reference velocity. Also, note that no position is associated with the reference velocity; the position of the centroid and of each pursuing vehicle will be dealt with in later sections.

A. Matching a Constant Reference Velocity

Theorem 4.1 (Matching a Constant Reference Velocity): For a constant $\vec{r}_{\text{ref}}$, with magnitude $\|\vec{r}_{\text{ref}}\| \in [0,1]$, the steering controls
\[
\begin{align*}
u_k &= -K \vec{n}_k^T (\vec{t} - \vec{r}_{\text{ref}}) \\
v_k &= -K \vec{b}_k^T (\vec{t} - \vec{r}_{\text{ref}}),
\end{align*}
\] (14)

for all $k \in \{1, \ldots, N\}$, will asymptotically stabilize the velocity of the centroid, $\vec{t}$, to the reference velocity, $\vec{r}_{\text{ref}}$, provided all vehicle headings are not initially parallel (i.e. $\text{rank}\{[t_1(0) \ t_2(0) \ \ldots \ t_N(0)]\} > 1$). Here, $K > 0$ is a control gain.

Proof: The proof follows from LaSalle’s Invariance Principle. Consider the candidate Lyapunov function
\[
V(\vec{g}) = \frac{N}{2} (\vec{t} - \vec{r}_{\text{ref}})^T (\vec{t} - \vec{r}_{\text{ref}}), \quad \vec{g} \in SO(3)^N
\] (15)
which is the order potential relative to the reference velocity. This candidate function is zero when the centroid velocity matches the reference velocity, $\vec{t} = \vec{r}_{\text{ref}}$, and positive otherwise. The derivative of (15) with respect to time is
\[
\dot{V}(\vec{g}) = N (\vec{t} - \vec{r}_{\text{ref}})^T \dot{\vec{t}}
\] (16)
because the reference velocity is constant. From the definition of the centroid velocity and (1),
\[
\dot{\vec{t}} = \frac{1}{N} \sum_{k=1}^N \vec{t}_k = \frac{1}{N} \sum_{k=1}^N (u_k \vec{n}_k + v_k \vec{b}_k).
\] (17)

Noting that the control inputs (14) come from a gradient along the candidate Lyapunov function,
\[
\dot{\vec{t}} = u_k \vec{n}_k + v_k \vec{b}_k = -K \frac{\partial V(\vec{t})}{\partial t_k} = -K (\vec{t} - \vec{r}_{\text{ref}}).
\] (18)
The controls, \( u_k \) and \( v_k \), can be isolated using inner products of \( \dot{t}_k \) with \( n_k \) and \( b_k \), respectively. Substituting these controls with (17) in (16) yields
\[
\dot{V}(g) = -K(T \cdot \dot{r}_{ref})^T \sum_{k=1}^{N} (n_k^T(\dot{t} - \dot{r}_{ref})n_k \\
+ b_k^T(\dot{t} - \dot{r}_{ref})b_k) \\
= -K \sum_{k=1}^{N} (||n_k^T(\dot{t} - \dot{r}_{ref})||^2 + ||b_k^T(\dot{t} - \dot{r}_{ref})||^2).
\]
(19)

The time derivative of the candidate Lyapunov function (20) is negative-semidefinite for \( K > 0 \). Because \( SO(3)^N \) is a closed and bounded set, LaSalle’s Invariance Principle states that all solutions converge to the largest invariant set contained in
\[
E = \left\{ g \left| ||n_k^T(\dot{t} - \dot{r}_{ref})||^2 + ||b_k^T(\dot{t} - \dot{r}_{ref})||^2 = 0 \text{ } \forall k \right. \right\}.
\]
(20)

All points in this set are invariant with respect to the controlled dynamics because both \( u_k \) and \( v_k \) are identically zero for all points in \( E \).

The set \( E \) contains both stable and unstable equilibria. The equilibria characterized by \( \dot{t} = \dot{r}_{ref} \) are stable due to the gradient-based design of the controller. The remaining equilibria, for which \( \dot{t} \neq \dot{r}_{ref} \) and both \( n_k \) and \( b_k \) are perpendicular to \( \dot{t} - \dot{r}_{ref} \), are characterized by all vehicle headings being parallel to \( \dot{t} - \dot{r}_{ref} \). For these states, a heading perturbation always exists that will result in acceleration in the direction of the perturbation. Thus these states are unstable equilibria.

**Corollary 4.2 (Aligned Set Stability):** Stability to an aligned state in which all vehicles have the same heading occurs when \( \dot{r}_{ref} = 1 \).

**Corollary 4.3 (Balanced Set Stability):** Stability to a state in which the centroid of the group is fixed, termed the balanced set, occurs when \( \dot{r}_{ref} = 0 \).

### B. Matching a Known Dynamic Reference Velocity

Matching of the group centroid velocity to a dynamic reference velocity is achieved using feedforward control based on the acceleration of the target vehicle. If the target is a virtual vehicle, then this acceleration can be known perfectly. Otherwise, it must be estimated online, a topic outside the scope of the present paper. The approach taken here is to assume knowledge of the reference velocity and add an appropriate second term to both \( u_k \) and \( v_k \). For example, \( u_k \) becomes \( u_k^{const} + u_k^{dyn} \), where \( u_k^{const} \) is the control law devised in the previous section for matching a known constant reference velocity, and \( u_k^{dyn} \) is a new term to be derived in this section.

**Theorem 4.4 (Dynamic Reference Velocity Matching):** For a non-constant reference velocity \( \dot{r}_{ref} \in [0, 1] \), the controls
\[
\begin{align*}
  u_k &= -Kn_k^T(\dot{t} - \dot{r}_{ref}) + u_k^{dyn} \\
  v_k &= -Kb_k^T(\dot{t} - \dot{r}_{ref}) + v_k^{dyn}
\end{align*}
\]
(22)

where vectors \( u^{dyn} \) and \( v^{dyn} \) satisfy
\[
J \begin{bmatrix} u^{dyn} \\ v^{dyn} \end{bmatrix} = \dot{r}_{ref}
\]
(23)

and
\[
J = \begin{bmatrix} n_1 & n_2 & \ldots & n_N & b_1 & b_2 & \ldots & b_N \end{bmatrix}
\]
(24)

ensure asymptotic stability of the velocity of the group centroid, \( \dot{t} \), to the dynamic reference velocity, \( \dot{r}_{ref} \).

**Proof:** Consider the candidate Lyapunov function (14) used for constant reference velocity matching. This function is positive for all states in which the velocity of the centroid does not match the reference velocity and zero otherwise. Taking the time derivative of this function,
\[
\dot{V}(g) = N(\dot{t} - \dot{r}_{ref})^T(\dot{t} - \dot{r}_{ref}).
\]
(25)

Applying \( \dot{t} \) from (17),
\[
\dot{V}(g) = (\dot{t} - \dot{r}_{ref})^T \left( \sum_{k=1}^{N} (u_k n_k + v_k b_k) - \dot{r}_{ref} \right).
\]
(26)

Substituting the control law (22) gives
\[
\begin{align*}
\dot{V}(g) &= -K \sum_{k=1}^{N} (||n_k^T(\dot{t} - \dot{r}_{ref})||^2 + ||b_k^T(\dot{t} - \dot{r}_{ref})||^2) \\
&+ (\dot{t} - \dot{r}_{ref})^T \left( \sum_{k=1}^{N} (u_k^{dyn} n_k + v_k^{dyn} b_k) - \dot{r}_{ref} \right).
\end{align*}
\]
(27)

The first term is identical to the constant velocity matching result (20), and the second term is zero by construction (23). Note that (23) has a solution provided not all vehicles are parallel. Thus, the reasoning used for matching a constant reference velocity holds and the proof is complete.

Simulation results are given in Fig. 2.

![Fig. 2. A group of N = 5 vehicles steers to make the centroid velocity match a reference velocity with dynamics similar to (1) but with variable speed. The target both turns and increases speed linearly from 0.3 to 0.82. A gain of K = −0.15 and random initial conditions were used to produce this result. The corresponding Lyapunov function (15) decreases exponentially from 3.0e⁻² to 5.5e⁻⁴ over the 25sec simulation period.](image-url)
V. Outer Loop Reference Velocity Control

The purpose of the outer loop controller for each vehicle is to automatically generate the reference velocity in such a way as to regulate spatial errors between the group centroid and the target vehicle and to compensate for inaccuracies in the feedforward control. Conceptually, the controller developed here is similar to previous work [9], [15] in that the reference velocity is directed towards the target when the spatial error is large, and matches the target velocity when there is no spatial error.

The transition between spatial error reduction and velocity matching is achieved smoothly by a weighting function, \( \omega(\rho) \). Here, \( \rho = \| r_t - \bar{r} \| \) is the distance between the group centroid and the target vehicle. The weighting function determines the commanded centroid velocity as

\[
\dot{r}_{\text{ref}} = (1 - \omega(\rho))\dot{r}_t + \omega(\rho)(r_t - \bar{r})/\rho,
\]

where \( \lim_{\rho \to 0} \omega(\rho)/\rho = 0 \). A proof that this outer loop controller takes the centroid to the target assuming the velocity of the group centroid matches the reference velocity was presented for the planar case in [9], [15]. An identical proof works in three dimensions and is omitted due to space constraints.

As an example, consider \( \omega(\rho) = 1 - e^{-\alpha \rho} \). This weighting function exponentially decays from zero to one as \( \rho \) increases from zero to infinity, representing a smooth transition from velocity matching to spatial error regulation. In this case the outer loop reference command can be computed as

\[
\dot{r}_{\text{ref}} = e^{-\alpha \rho}\dot{r}_t + (1 - e^{-\alpha \rho})(r_t - \bar{r})/\rho.
\]

A feedforward control term for each individual can be computed similarly to (23), but based on the outer loop reference command. The results of Theorem 4.4 indicate that the group centroid velocity will converge to the reference velocity. In turn, the reference velocity will drive the collective centroid to the target.

VI. Spacing Control

The control laws designed thus far stabilize the velocity of the group centroid to a possibly dynamic reference velocity without regard for the physical locations of the individual vehicles. In fact, once the centroid velocity has stabilized to the reference velocity, the steering control on each vehicle decreases to zero, and each vehicle proceeds along its current direction out to infinity. The objective of this section is to modify the steering controller in such a way as to preserve the previously developed velocity matching but also to add a component that causes individual vehicles to stay near the centroid. This additional term is referred to as the spacing control, and the total steering control will be the velocity matching component (vm) plus the new spacing component. Specifically, \( u = u_{\text{vm}} + u_{\text{space}} \) and \( v = v_{\text{vm}} + v_{\text{space}} \).

This idea stems from the fact that the velocity matching portion of the control has unused degrees of freedom. The spacing control should be restricted to these unused degrees of freedom and not change the velocity of the centroid. Thus, permissible spacing controls satisfy

\[
\frac{1}{N} \sum_k (u_k^{\text{space}} n_k + v_k^{\text{space}} b_k) = 0.
\]

Writing the above equation as a matrix, valid spacing controls must lie in the null space of \( J \) from (24),

\[
\begin{bmatrix} u^{\text{space}} \\ v^{\text{space}} \end{bmatrix} \in \text{Null}\{J\}.
\]

Any vector \( w \in \mathbb{R}^{2N} \) can be projected to \( \tilde{w} \) which is in the Null of \( J \) using the orthogonal projection

\[
\tilde{w} = Pw \quad \text{where} \quad P \equiv I - pp^T,
\]

and \( p \) is an orthogonal basis for the range of \( J^T \).

Theorem 6.1 (Combined matching and spacing): Any spacing control that satisfies (31) will not affect velocity matching.

Proof: Using the Lyapunov candidate from (15), the time derivative is unchanged from (26). Applying the control,

\[
\begin{align*}
\dot{u}_k &= -Kn_k^T(\hat{t} - \hat{r}_{\text{ref}}) + u_k^{\text{dyn}} + u_k^{\text{space}} \\
\dot{v}_k &= -Kv_k^T(\hat{t} - \hat{r}_{\text{ref}}) + v_k^{\text{dyn}} + v_k^{\text{space}},
\end{align*}
\]

where the \( \text{dyn} \) term satisfies (23) as before, the result is (20) assuming the spacing control null space construction (31).

Not all controls that satisfy condition (31) will result in vehicles staying near the centroid; controls that satisfy this condition simply do not interfere with velocity matching. Ensuing in this section is the development of a specific spacing control law that satisfies (31) and results in helical vehicle trajectories.

A. Helical Spacing Control

The specific spacing control proposed here is intended to keep individuals near the centroid by making them perform helices of a common radius with instantaneous center of oscillation (ICO) located at the collective centroid. The spacing controller is designed for a constant reference velocity, but performs well even for dynamic reference velocities because it acts on a faster time scale than velocity matching. A simulation result from the control law developed in this section is shown in Fig. 3.

The ideas behind the helix-forming spacing control are simple, however rigorous analysis is complicated by the spacing control constraint (30) in combination with the constant-speed nonholonomic vehicle model. Overall, the control design is based on the orientation of the reference velocity. To avoid singularities when the reference velocity, \( \hat{r}_{\text{ref}} \), is zero, its magnitude, \( s_{\text{ref}} \in [0, 1) \), is defined separately from its orientation, \( \hat{t}_{\text{ref}} \in \mathbb{R}^3 \), which is a unit vector.

The dynamics (1) ensure that the acceleration of each vehicle is perpendicular to its velocity, \( \mathbf{t}_k \). Thus, instead of working in terms of \( u_k \) and \( b_k \), a more intuitive approach is to prescribe the acceleration as a vector, \( q_k \), perpendicular
to \( t_k \). The control inputs \( u_k \) and \( v_k \) come from the inner products

\[
  u_k = n_k^T q_k \quad \text{and} \quad v_k = b_k^T q_k. \tag{34}
\]

Note that if \( q_k \) is not perpendicular to \( t_k \), the applied acceleration will be the projection of \( q_k \) into the plane perpendicular to \( t_k \).

The spacing control is composed of four terms: helix, beacon, speed, and plane. The net desired acceleration for each vehicle, \( k \), is written as the vector

\[
  q_k = K_{\text{helix}}^{} q_k^{\text{helix}} + K_{\text{beacon}}^{} q_k^{\text{beacon}} + K_{\text{speed}}^{} q_k^{\text{speed}} + K_{\text{plane}}^{} q_k^{\text{plane}}, \tag{35}
\]

where all control gains are positive. The desired vector of controls may not satisfy the spacing control constraint (30) and thus must be projected using \( P \) from (32).

The Helix Term: The helix term causes a rotation of all vehicle headings about the reference velocity. The acceleration for each vehicle necessary to produce this motion is

\[
  q_k^{\text{helix}} = \hat{t}_{\text{ref}} \times t_k. \tag{36}
\]

In steady state, all vehicles have a common natural frequency of \( \omega = K_{\text{helix}}^{} \), and the per-vehicle radius is \( \|t_k \times \hat{t}_{\text{ref}}\|/(K_{\text{helix}}^{}) \).

The Plane Term: The only component missing at this point is that some vehicles may be far behind the centroid while others are far ahead of it. The plane term corrects this problem with the proportional vector:

\[
  q_k^{\text{plane}} = \hat{t}_{\text{ref}}^T (\bar{r} - r_k) \hat{t}_{\text{ref}}. \tag{44}
\]

The result is that all vehicles are stabilized to the plane perpendicular to the centroid velocity, containing the centroid.

Discussion: The resulting control will not disturb the action of velocity matching because the constraint (30) is satisfied by projection. However, this projection leads to complications in formulating a proof of the results. In simulation, the proposed control law almost always works as indicated. The one exception occurs when the speed of the reference velocity is zero. In this case all terms except the beacon term settle quickly. The result is that all vehicles stabilize to circular trajectories in the plane perpendicular to the reference orientation containing the centroid. The ICOs, however, do not always stabilize to the centroid for this particular case. This is an issue that will be examined in future work.
VII. Conclusion

The results of this paper are a direct extension to three dimensions of previous work on planar target tracking using oscillator models. The extension requires rewriting the standard oscillator model in a form appropriate for Frenet Natural Frames. The controller was designed incrementally and offers the guarantee, under some assumptions, that the group centroid will track the target vehicle. Spacing control based on helices was added, and simulation results were presented.

Many possible directions are available for extensions to this work in the future. Currently, all vehicles are assumed to have the correct state information about all other vehicles and about the target vehicle. A more realistic setting would include state estimators supplied with discrete time communication events that induce topological constraints as well as noise and delay in transmitted data. Some recent work has looked at the planar phase coupled oscillator in discrete time [17]. An interesting question is whether the results found there apply to the three-dimensional setting, in which case a more realistic communication model could be studied.

Another area of possible future work is formalizing the helical spacing control, which was presented without proof in the current work. The proposed control clearly has some undesirable equilibria, the stability of which may become clear through a Lie group analysis.

Finally, a hardware demonstration of the proposed control laws would be instructive. Currently available the the University of Washington is a testbed consisting of three free-swimming robotic fish and an instrumented water tank. The vehicles are well-modeled by Frenet Natural Frames and are capable of performing the proposed control.

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References