

Stability of Time-delay Feedback Switched Linear Systems

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Abstract—We address stability of state feedback switched linear systems in which delays are present in both the feedback state and the switching signal of the switched controller. For switched systems with average dwell-time switching signals, we provide a condition, in terms of upper bounds on the delays and in terms of a lower bound on the average dwell-time, to guarantee asymptotic stability of the closed loop. The condition also implies that, in general, feedback switched linear systems are robust with respect to small state delays and small switching delays. Our approach combines existing multiple Lyapunov function techniques with the merging switching signal technique, which gives relationships between the average dwell times of two mismatched switching signals and their mismatched times, in a novel and non-trivial way. We illustrate an application of our stability results for feedback switched systems with delays to the analysis of multi-agent dynamic networks with a consensus protocol in the presence of asymmetric delays and switching topologies. A methodology for numerical calculation based on LMIs is also included.

I. INTRODUCTION

Switched systems are dynamical systems represented by a family of subsystems and a switching signal that indicates the active subsystem at every time. Such classes of dynamical systems, which can also be seen as higher level abstractions of hybrid systems obtained by neglecting the switching mechanisms, can be found in many physical plants (e.g., multi-modal aircrafts [1], walking robots [2]) and modern control design (e.g., gain scheduling [3], supervisory control [4]); see, for example, [5], [6] for further background and references on switched systems. In this paper, we address stability of a particular type of switched systems, namely *feedback switched systems*, in the presence of delays. By a feedback switched system, we mean a switched plant connected in closed loop with a switched feedback controller.

In the ideal case, the controller has instant access to both the plant's state and switching signal. In such a case, the controller's switching and the plant's switching are synchronized, and the closed loop can be represented by a single switched system, for which various tools for stability analysis are available (e.g., [7], [8]). However, when the information available to the controller is a delayed version of that at the plant (for example, when the plant and the controller communicate via a communication channel), the closed loop system can have *asynchronous switching signals* (those from the plant and those from the controller) as well as *delayed states* (in the controller). Finding conditions on the systems

and the delays to guarantee stability of the closed loop under such delays is thus of interest in control research.

Stability of switched systems with delays is a control systems research topic that has only recently been treated ([9], [10], [11], [12]). All of the foregoing references assume state delays only and no switching delays. The work [13] has considered stabilization of feedback switched systems with switching delays but without state delays, and the result is restricted to dwell-time switching. In this work, we consider both state delays and switching delays as well as more general classes of switching signals (average dwell-time switching signals).

The contribution of our work is two-fold. First, we provide an explicit condition in terms of upper bounds on the delays and lower bounds on the average dwell-time of the switching signal to guarantee asymptotic stability of the closed-loop system. Second, our condition also implies that stability of feedback switched linear systems with average dwell-time switching is robust with respect to small delays in both the state and the switching signal, provided that the average dwell-time is large enough. We further show that the robustness property of switched feedback with delays still holds in the presence of small additive unmodeled dynamics.

Another contributions of our work, from the perspective of tools for switched system analysis, are to provide a merging switching signal technique to deal with two switching signals in a closed loop and to provide a multiple Lyapunov functions technique for analyzing stability of switched feedback systems with state delays, building upon the technique to deal with delays in [14], the small-gain technique in [14], [15], the multiple Lyapunov function technique for switched systems (without feedback) [7] and the average dwell-time switching concept [7]. Such a Lyapunov function approach provides a unified framework to treat both delays and unmodeled dynamics [14], and the approach can be readily extended to systems with disturbances and to nonlinear systems [14], [15].

The type of feedback switched systems with delays described here could find application, for example, in consensus networks (see Section V), or in multi-modal control systems where controller selection takes a certain minimum amount of time (e.g., when controller selection is carried out by human operators [16]).

The organization of the paper is as follows: In Section II, we discuss motivations for studying feedback switched systems with delays and review the literature. We then mathematically formulate the stability problem for such feedback switched systems with both state delays and switching delays in Section III. The main result is presented in Section IV. We illustrate our result for consensus networks with switching topology and

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communication delay in Section V. In section VI, we provide an LMI approach for numerical calculation of the quantities in the main theorem, accompanied by a numerical example. Conclusions and discussion of future work are given in Section VII.

II. BACKGROUND

A. Switched systems

A switched linear control system is of the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is the switching signal mapping time to some finite index set \mathcal{P} , and $A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times m}$, $p \in \mathcal{P}$, are the state and input matrices. The *switching signal* σ is a piecewise-constant continuous-from-the-right function taking values in the set \mathcal{P} . The discontinuities of σ are called *switching times* or *switches*. We assume that no jump occurs in the state at a switching time. We also assume that only finitely many switches can occur in any finite interval (i.e. no Zeno behavior). Thus, a solution of (1) in the Caratheodory sense is well-defined on $[0, \infty)$ and unique for every initial state $x(0)$ and piecewise-continuous input u (see, e.g., [17]).

B. Motivation for feedback switched systems with delays

We discuss two scenarios where delays come into feedback switched systems, one from multi-agent networks and the other from multi-modal control. The first example is a multi-agent network with switching topology and communication delay. Consider a network of n homogeneous agents whose dynamics are $\dot{x}_i = u_i$, where $x_i \in \mathbb{R}^{n_x}$ is the state of agent i , and u_i is the control input, $i = 1, \dots, n$. Each agent can communicate with only those agents currently within a finite neighborhood of its state as described by a graph (also called a communication topology in this context). Denote by \mathcal{N}_i^G the nodes (agents) that have connection with a node (agent) i of a graph G . Suppose that time τ_x is required to transmit a signal from one node to another node in the neighborhood (for simplicity, assume that τ_x is uniform and a constant). Then the information agent i received from agent j at time t is $x_j(t - \tau_x)$. Suppose that every agent is using the same control law (or protocol): $u_i(t) = g(x_i(t), \{x_j(t - \tau_x)\}_{j \in \mathcal{N}_i^G})$. Then the collective dynamics of the network are of the form $\dot{x}(t) = f_G(x(t), x(t - \tau_x))$ for some function f_G depending on the structure of G , where $x = (x_1, \dots, x_n)$. If f_G is separable in terms of $x(t)$ and $x(t - \tau_x)$ and the topology G is time-varying, then the collective dynamics can be written as a feedback switched system with state delays. For example, if $f_G(x(t), x(t - \tau_x)) = f_{1G}(x(t)) + f_{2G}(x(t - \tau_x))$ (a linear consensus protocol $u_i = \sum_{j \in \mathcal{N}_i} x_j(t - \tau_x) - x_i(t)$ leads to such a form) and $G : [0, \infty) \rightarrow \{G_1, \dots, G_m\}$, where $\{G_1, \dots, G_m\}$ is a finite collection of graphs, then the collective dynamics can be written as a feedback switched system with the switched plant $\dot{x} = f_{\sigma}(x, u)$ and the switched feedback controller $u = h_{\sigma}(x(t - \tau_x))$, where $f_p(x, u) = f_{1G_p}(x) + u$ and $h_p(x) = f_{2G_p}(x)$, $p = 1, \dots, m$.

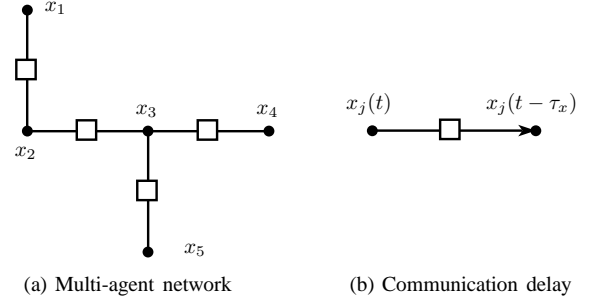


Fig. 1. Multi-agent network (a) with possible communication delays of the form (b).

The second example of feedback switched systems with delays arises in the context of multi-modal control. Consider a process \mathbf{P} (for example, a robot) having multiple operating modes to be controlled by a multi-controller \mathbf{C} (see Fig. 2). Suppose that \mathbf{P} is represented by a switched system $\dot{x} = \tilde{f}_{\sigma}(x, u)$, where u is the control input, and $\sigma : [0, \infty) \rightarrow \{1, \dots, m\}$ is the switching signal. Suppose that \mathbf{C} is a switched state feedback controller, $u = K_{\sigma}x$. Now, if the switching signal σ has to be sent to the controller \mathbf{C} over a communication channel with delay τ_s , then $\sigma_c(t) = \sigma(t - \tau_s)$, and the closed loop is a feedback switched system with delays. Another case is when the process of deciding which controller to switch to requires a certain amount of time τ_s . In such cases, even if no communication delay is present, $\sigma_c(t) = \sigma(t - \tau_s)$, and the closed loop is a feedback switched system with delays.

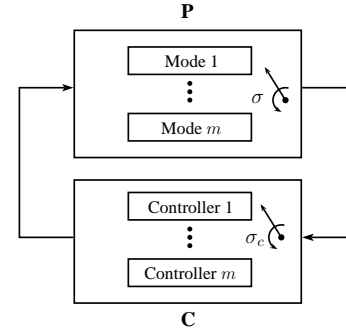


Fig. 2. Multi-modal plant and controller, where \mathbf{P} is the plant with the switching signal σ , and \mathbf{C} is a multi-controller with the switching signal σ_c .

C. Literature review

The vast majority of the results concerning feedback switched systems (e.g., [9], [10], [11], [12], [13], [18]) assume state delays only and do not consider switching delays, i.e., the closed loop is a switched system of the form $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma}x(t - \tau_x)$. We found the work [13] that has considered stabilization of feedback switched systems with switching delay, but no state delay was considered, and the result is restricted to dwell-time switching. For switched systems with state delay and without switching delay, stability results have been obtained ([9], [10]) for the class of average dwell-time switching signals (as defined in [7]). The techniques to

deal with switched systems with state delays are multiple Lyapunov-Krasovskii functionals ([9], [12]), or Lyapunov-Razumikhin functions [10]—these methods are extensions of the well-known Lyapunov techniques for nonswitched linear systems (see, e.g., [19], [20]) to switched systems. In [18], common Lyapunov-Krasovskii functionals are used (together with the technique in [21]) to design switching surfaces to guarantee stability for feedback switched systems with state delays.

Remaining open is stability of feedback switched systems with both state and switching delays, and this problem is far from a trivial extension of the case with only state delays because the closed loop has two asynchronous switching signals (one is from the plant, and one is from the controller). We will approach this problem by using a merging switching signal technique to deal with the two switching signals in the closed loop, and by combining the multiple Lyapunov function technique, the technique to deal with delays in [14] (which is built upon [15]), the small-gain technique (in [14], [15]), and the average dwell-time switching concept in [7]. Such a Lyapunov function approach provides a unified framework to treat both state delays and switching delays as well as unmodeled dynamics, disturbances, and quantization [14], and the approach can be extended to nonlinear systems ([14], [15]).

III. PROBLEM FORMULATION

Consider the switched system (1).

Assumption 1 (A_p, B_p) are stabilizable $\forall p \in \mathcal{P}$.

Let K_p be matrices such that for every $p \in \mathcal{P}$, $A_p + B_p K_p$ is Hurwitz. The ideal switched state feedback controller is

$$u = K_\sigma x, \quad (2)$$

and the closed loop is $\dot{x} = (A_\sigma + B_\sigma K_\sigma)x$. Recall [7] that a switching signal σ is an average dwell-time signal if the number of switches in any interval $[t_0, t)$, denoted by $N_\sigma(t, t_0)$, satisfies

$$N_\sigma(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a} \quad t \geq t_0$$

for some constant $N_0 \geq 1$; N_0 is called a *chatter bound*. Denote by $\mathcal{S}_{\text{ave}}[\tau_a, N_0]$ the class of switching signals with average dwell-time τ_a and chatter bound N_0 . Note that when $N_0 = 1$, we have switching signals with dwell-time τ_a because no more than one switch can occur in any interval of length less than τ_a . As in [7], if the switching signal σ has an average dwell-time τ_a , then there exists a positive number τ_a^* , which depends on $A_p + B_p K_p$, $p \in \mathcal{P}$, such that the switched system $\dot{x} = (A_\sigma + B_\sigma K_\sigma)x$ is asymptotically stable for all $\tau_a \geq \tau_a^*$.

Now, suppose that a delay τ_x exists between the plant's output (assumed to be the exact state in this paper) and the state available to the controller for some constant $\tau_x > 0$, i.e. the controller receives the state $x(t - \tau_x)$ at time t . Because the controller is also switching, two scenarios are possible: 1) the switching signal available to the controller is synchronized with the switching signal σ of the plant, or 2) the controller's

switching signal is a delayed version of the plant's switching signal. The first case could happen, for example, when the switching signal is generated by timing, and the plant and the controller use the same clock. The second case occurs, for example, when information about the switching signal of the plant has to be sent to the controller over a communication channel (see Fig. 3a). In the first case, the control signal going into the plant is $u(t) = K_{\sigma(t)}x(t - \tau_x)$, and in the second case, $u(t) = K_{\sigma(t - \tau_s)}x(t - \tau_x)$. In the case of input delays (see Fig. 3b), the control signal going into the plant is $u(t) = K_{\sigma(t - \tau_d)}x(t - \tau_d)$.

In all of the above cases, the control signal going into the plant is of the form

$$u(t) = K_{\sigma(t - \tau_s)}x(t - \tau_x) \quad (3)$$

for some non-negative constants τ_s and τ_x . The formula (3) also covers the case where input delay, output delay, and switching delay are all present (i.e. superimposing Fig. 3a on Fig. 3b), in which case $u(t) = K_{\sigma(t - \tau_s - \tau_d)}x(t - \tau_x - \tau_d)$.

Given the closed loop system consisting of (1) and (3), for every initial data $x : [-\tau_x, 0] \rightarrow \mathbb{R}^n$, piecewise continuous input $u : [0, \infty) \rightarrow \mathbb{R}^m$, and switching signal $\sigma : [-\tau_x, \infty) \rightarrow \mathcal{P}$, a solution (or trajectory) x exists for all time in $[0, \infty)$ and is unique in the Caratheodory sense (in view of the linearity of the system).

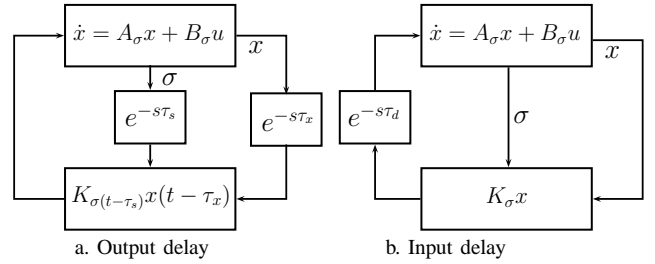


Fig. 3. Feedback switched systems with delays

For $\tau_x = \tau_s = 0$, we have asymptotic stability of the closed-loop switched system under average dwell-time switching, and for large enough τ_x with $\tau_s = 0$, we could get instability because large enough state delays may cause instability for every subsystem of a switched system¹. Intuitively, for $\tau_s = 0$, one could expect that an upper bound on τ_x exists that guarantees closed-loop stability (although, rigorously, even the existence of such a bound is not apparent for switched systems and has to be proved). Stability of the closed loop is even more challenging when switching delays are present, i.e. $\tau_s \neq 0$, and the existence of bounds on both τ_s and τ_x for closed loop stability is not obvious. The main problem is to determine if such bounds on τ_x and τ_s indeed exist and for which type of switched systems, and to quantify such bounds if they exist.

Problem: For the switched system (1) with the control (3), find conditions on τ_x and τ_s and determine classes of switching

¹An upper bound on the delays for stability is a sufficient condition only; it is possible to have stable non-switched linear systems with arbitrarily large delays.

signal σ to guarantee asymptotic stability of the closed loop system.

IV. MAIN RESULT

Our main theorem characterizes the relationship between the delays and the average dwell-time of the plant's switching signal to guarantee asymptotic stability. Before presenting the theorem, we define some notation and variables.

A. Notation

Let $\bar{A}_{p,q} := A_p + B_p K_q$, and let $\mathcal{P}^2 := \mathcal{P} \times \mathcal{P}$. Let \mathcal{P}_s^2 be the set of all indices in \mathcal{P}^2 such that \bar{A}_i is Hurwitz for all $i \in \mathcal{P}_s^2$. Note that the set \mathcal{P}_s^2 contains at least all elements of the form (p, p) for all $p \in \mathcal{P}$ in view of the fact that $\bar{A}_{p,p}$ are Hurwitz by construction. Let $\mathcal{P}_u^2 := \mathcal{P}^2 \setminus \mathcal{P}_s^2$.

1) *The rates λ_s and λ_u and the gain γ* : From the Lyapunov theory for linear systems, there always exist positive definite quadratic functions $V_j : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$\frac{\partial V_j}{\partial x}(\bar{A}_j x + v) \leq -\lambda_s V_j + \gamma |v|^2 \quad \forall v \in \mathbb{R}^n, j \in \mathcal{P}_s^2 \quad (4)$$

for some positive constants λ_s and γ (note that $j \in \mathcal{P}_s^2$ is a pair of indices), where $|\cdot|$ denotes the 2-norm of vectors. For example, one can have $V_j(x) = x^T P_j x$, where P_j is the solution to the Lyapunov equation $(\bar{A}_j + \frac{\lambda_s}{2} I)^T P_j + P_j (\bar{A}_j + \frac{\lambda_s}{2} I) = -I$, and $\lambda_s/2$ is the smallest real part of the eigenvalues of $\bar{A}_j, j \in \mathcal{P}_s^2$ with $0 < \lambda_s < \bar{\lambda}_s$. The existence of a common λ_s and γ follows from the fact that \mathcal{P} is finite. Also, there always exist positive definite quadratic functions $V_j : \mathbb{R}^n \rightarrow [0, \infty)$ such that

$$\frac{\partial V_j}{\partial x}(\bar{A}_j x + v) \leq \lambda_u V_j + \gamma |v|^2 \quad \forall v \in \mathbb{R}^n, j \in \mathcal{P}_u^2 \quad (5)$$

for some positive constants λ_u and γ (for any matrix \bar{A}_j , one can always pick a positive definite matrix P_j and a large enough $\alpha > 0$ such that $\bar{A}_j^T P_j + P_j \bar{A}_j \leq \alpha P_j$). Again, the existence of a common γ in (4) and (5) follows from the fact that \mathcal{P} is finite. The constant λ_s is the decay rate, the constant λ_u is the growth rate, and γ is the input gain for the family of systems $\dot{x} = \bar{A}_i x + v, i \in \mathcal{P}^2$.

2) *The constants α_1, α_2 , and μ* : Because V_j are quadratic positive definite, there exist $\alpha_1, \alpha_2 > 0$, and $\mu \geq 1$ such that

$$\alpha_1 |x|^2 \leq V_j(x) \leq \alpha_2 |x|^2 \quad \forall j \in \mathcal{P}^2 \quad (6a)$$

$$V_j(x) \leq \mu V_i(x) \quad \forall i, j \in \mathcal{P}^2. \quad (6b)$$

One can always set $\mu = \alpha_2/\alpha_1$, but a smaller μ may exist, depending on the particular V_j .

3) *The constants c_1 and c_B* : Define $c_B := \sup_{(p,q) \in \mathcal{P}^2} \|B_p K_q\|$ and $c_A := \sup_{p \in \mathcal{P}} \|A_p\|$, where $\|\cdot\|$ denotes the induced matrix norm with $|\cdot|$ vector norm. Let $c_1 := c_A + c_B$. The constant c_A is the maximum norm of the system matrices, and c_B is the maximum gain of the feedback state coming into the dynamics.

B. Main result

Let $x_d(t) := \|x\|_{[t-2\tau_x, t]}$.

Theorem 1 Consider the switched system (1) with the controller (3). Suppose that Assumption 1 holds and that $\sigma \in \mathcal{S}_{ave}[\tau_a, N_0]$ for some positive constants τ_a and N_0 . Let

$$\begin{aligned} \bar{\tau}_a &:= \frac{\tau_a}{2} \\ \bar{N}_0 &:= 2N_0 + \frac{\tau_s}{\tau_a}. \end{aligned} \quad (7)$$

If

$$(\tau_x c_1 c_B)^2 \kappa + \frac{\ln \mu}{\bar{\tau}_a} < \lambda_s - \frac{\tau_s}{\tau_a} (\lambda_s + \lambda_u), \quad (8)$$

where $\kappa := \mu^{\bar{N}_0} \exp((\lambda_s + \lambda_u) N_0 \tau_s) (\alpha_2 \gamma / \alpha_1)$, then we have

$$|x_d(t)|^2 \leq \left(g_0 e^{2\lambda' \tau_x} e^{-\lambda'(t-t_0)} + g_1(\tau_x) \right) |x_d(t_0)|^2 \quad (9)$$

for all $t \geq t_0 \geq \tau_x$ for some constants $\{\lambda', g_0\} > 0$ and function $g_1 : [0, \infty) \rightarrow [0, \infty)$ such that $g_1(\tau_x) \rightarrow 0$ as $\tau_x \rightarrow 0$.

A key technique in the proof of Theorem 1 is the merging switching signal technique. To deal with mismatched switching signals, such as σ and σ_c , the idea is to create a (virtual) new switching signal $\sigma' : [0, \infty) \rightarrow \mathcal{P} \times \mathcal{P}$ as follows:

$$\sigma'(t) := (\sigma(t), \sigma_c(t)). \quad (10)$$

The merging action is denoted by \oplus such that $\sigma' = \sigma \oplus \sigma_c$. From the definition, the set of switching times of $\sigma' = \sigma \oplus \sigma_c$ clearly is the union of the sets of switching times of σ and of σ_c (see Fig. 4 for illustration).

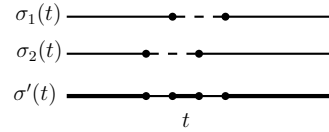


Fig. 4. Switching signals. The dots are the switching times, the dashed lines and solid lines of $\sigma_1(t)$ and $\sigma_2(t)$ are the two modes (or indices), and the thick lines (thin lines) of $\sigma'(t)$ are when $\sigma_1(t)$ and $\sigma_2(t)$ match (do not match).

While the idea of merging switching signals is rather straightforward, what is nontrivial and interesting is the question of determining properties of the merged switching signal from properties of the two switching signals before merging.

Lemma 1 Let $\sigma_1 \in \mathcal{S}_{ave}[\tau_{a_1}, N_1]$ and $\sigma_2 \in \mathcal{S}_{ave}[\tau_{a_2}, N_2]$. If $\sigma_1 = \sigma_2$, then

$$\sigma_1 \oplus \sigma_2 \in \mathcal{S}_{ave}[\min\{\tau_{a_1}, \tau_{a_2}\}, \max\{N_1, N_2\}],$$

otherwise $\sigma_1 \oplus \sigma_2 \in \mathcal{S}_{ave}[\tau_a, N_1 + N_2]$, where $\tau_a := (1/\tau_{a_1} + 1/\tau_{a_2})^{-1}$.

Proof: If $\sigma_1 = \sigma_2$, i.e. $\sigma_1(t) = \sigma_2(t) \forall t$, then clearly $\sigma' := \sigma_1 \oplus \sigma_2 = \sigma_1 = \sigma_2$ in view of the definition of \oplus .

From the definition of average dwell-time, we have, for an arbitrary interval $[t_0, t)$, $N_{\sigma'}(t, t_0) \leq N_1 + (t - t_0)/\tau_{a_1}$ and $N_{\sigma'}(t, t_0) \leq N_2 + (t - t_0)/\tau_{a_2}$, which implies $N_{\sigma'}(t, t_0) \leq \max\{N_1, N_2\} + (t - t_0)/\min\{\tau_{a_1}, \tau_{a_2}\}$, and the claim in the lemma follows.

Suppose that $\sigma_1 \neq \sigma_2$. Consider an arbitrary interval $[t_0, t)$. Denote by n_1 the number of switches of σ_1 in $[t_0, t)$, and by n_2 the number of switches of σ_2 in $[t_0, t)$. Then the number of switches of $\sigma_1 \oplus \sigma_2$ in $[t_0, t)$ is less than or equal to $n_1 + n_2$. Because $\sigma_1 \in \mathcal{S}_{\text{ave}}[\tau_{a_1}, N_1]$, we have $n_1 \leq N_1 + (t - t_0)/\tau_{a_1}$. Similarly, $n_2 \leq N_2 + (t - t_0)/\tau_{a_2}$. Therefore, $n_1 + n_2 \leq N_1 + N_2 + (t - t_0)(1/\tau_{a_1} + 1/\tau_{a_2})$, which implies $\sigma_1 \oplus \sigma_2 \in \mathcal{S}_{\text{ave}}[\tau_a, N_1 + N_2]$, where $\tau_a := (1/\tau_{a_1} + 1/\tau_{a_2})^{-1}$. ■

In the case where $\sigma_c = \sigma(t - \tau_s)$ for some constant τ_s , if $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$, then $\sigma_c \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$. For the case of time varying τ_s , if $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$, the characterization of σ_c , where $\sigma_c(t) = \sigma(t - \tau_s(t))$, is far from obvious and is stated in the lemma below. Whenever τ_s is time-varying, from here onward, we assume that the delayed switching signal $\sigma(t - \tau_s(t))$ is causal, meaning that the ordering of the switching times of $\sigma(t - \tau_s(t))$ is the same as the ordering of the corresponding switching times of σ .

Lemma 2 *Let $\sigma_1 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$ and $\sigma_2(t) := \sigma_1(t - \tau_s(t))$ for some function $\tau_s : [0, \infty) \rightarrow [0, \bar{\tau}_s]$. Then $\sigma_2 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0 + \bar{\tau}_s/\tau_a]$.*

Proof: Consider an arbitrary interval $[t_0, t)$. Let $\tau_k, k = 1, \dots, N$ be the switching times of σ_2 in $[t_0, t)$, where N is the number of switches in $[t_0, t)$, and let $\tau'_k, k = 1, \dots, N$ be the switching times of σ_1 corresponding to τ_k , i.e. $\tau_k = \tau'_k + \tau_s(\tau'_k)$.

Because $0 \leq \tau_s(r) \leq \bar{\tau}_s \forall r$, we have $\tau'_N \leq \tau_N < t$ and $\tau'_1 \geq \tau_1 - \bar{\tau}_s \geq t_0 - \bar{\tau}_s$. We then have

$$\tau'_N - \tau'_1 < t - t_0 + \bar{\tau}_s. \quad (11)$$

Pick $\epsilon > 0$ arbitrarily small such that

$$\tau'_N + \epsilon - \tau'_1 \leq t - t_0 + \bar{\tau}_s. \quad (12)$$

Because there are N switches in $[\tau'_1, \tau'_N + \epsilon)$, and $\sigma_1 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$, we have

$$N \leq N_0 + (\tau'_N + \epsilon - \tau'_1)/\tau_a. \quad (13)$$

From (12) and (13), we get $N \leq N_0 + \bar{\tau}_s/\tau_a + (t - t_0)/\tau_a$, and hence, $\sigma_2 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0 + \bar{\tau}_s/\tau_a]$. ■

Lemma 2 says that a time-varying delayed version of an average dwell-time switching signal is also an average dwell-time switching signal with the same average dwell-time but with an increased chatter bound.

The following lemma characterizes the relationship between the total matching time and the total non-matching time when merging an average dwell-time signal with a delayed version of it.

Lemma 3 *Let $\sigma_1 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$ and $\sigma_2(t) := \sigma_1(t - \tau_s(t))$ for some positive function τ_s . For an interval (t_0, t) , let $m_{t_0, t}$ be the total time for which $\sigma_1(t) = \sigma_2(t)$, and let $\bar{m}_{t_0, t} := t - t_0 - m_{t_0, t}$. Suppose that $\tau_s(t) \leq \bar{\tau}_s$ for all t . If*

$$\bar{\tau}_s(\lambda_m + \lambda_{\bar{m}}) \leq (\lambda_m - \lambda)\tau_a \quad (14)$$

for some positive constants $\lambda_m, \lambda_{\bar{m}}$, and $\lambda \in [0, \lambda_m]$, then

$$-\lambda_m m_{t_0, t} + \lambda_{\bar{m}} \bar{m}_{t_0, t} \leq c_T - \lambda(t - t_0) \quad \forall t \geq t_0, \quad (15)$$

where $c_T := (\lambda_m + \lambda_{\bar{m}})N_0\bar{\tau}_s$.

Proof: Let τ_1, \dots, τ_N be the switching times of σ_1 in (t_0, T) with the convention $\tau_0 = t_0$ and $\tau_{N+1} = T$. If $\tau_{k+1} - \tau_k > \bar{\tau}_s$, then the total time in $[\tau_k, \tau_{k+1})$ for which $\sigma_1(t) \neq \sigma_2(t)$ will be at most $\bar{\tau}_s$ because the delay is bounded by $\bar{\tau}_s$. If $\tau_{k+1} - \tau_k \leq \bar{\tau}_s$, the total time in $[\tau_k, \tau_{k+1})$ for which $\sigma_1(t) \neq \sigma_2(t)$ will also be at most $\bar{\tau}_s$. Therefore, the total time in (t_0, T) for which $\sigma_1(t) \neq \sigma_2(t)$ will be at most $(N + 1)\bar{\tau}_s$, which means that

$$\bar{m}_{t_0, t} \leq (N + 1)\bar{\tau}_s, \quad (16)$$

and so

$$m_{t_0, t} \geq t - t_0 - (N + 1)\bar{\tau}_s. \quad (17)$$

Let $\kappa := \tau_a/\bar{\tau}_s$. Because $\sigma_1 \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$, and $N + 1 \leq N_{\sigma}(t_0, t)$, we have $N + 1 \leq N_0 + (t - t_0)/\tau_a$. Therefore,

$$(N + 1)\bar{\tau}_s \leq N_0\bar{\tau}_s + (t - t_0)/\kappa, \quad (18)$$

and, hence,

$$t - t_0 - (N + 1)\bar{\tau}_s \geq (1 - 1/\kappa)(t - t_0) - N_0\bar{\tau}_s. \quad (19)$$

From (16), (17), (18), and (19), we then have $-\lambda_m m_{t_0, t} + \lambda_{\bar{m}} \bar{m}_{t_0, t} \leq (\lambda_m + \lambda_{\bar{m}})N_0\bar{\tau}_s + (-\lambda_m(1 - 1/\kappa) + \lambda_{\bar{m}}/\kappa)(t - t_0)$. If (14) is true, then $-\lambda_m(1 - 1/\kappa) + \lambda_{\bar{m}}/\kappa \leq -\lambda$, and (15) follows. ■

Remark 1 *The condition (15) characterizes the relationship between the total matched time and the total unmatched time in any arbitrary interval for feedback switched systems with delays. Compared to the result involving the total time for stable and unstable modes of switched systems in [22], the lemma is novel in two aspects. First, it gives the relationship between the matched time and the unmatched time in terms of τ_x and t_s , whereas [22] starts with the relationship between the total time for stable and the total time for unstable modes as an assumption (and there is no delay in [22]). Second, the condition (15) has a nonzero term c_T on the right hand size, whereas in [22], this c_T term is zero. The term c_T relates to the overshoot when switching among stable and stable subsystems.*

Proof: [Theorem 1]

Before going into the details of the proof, we first outline the key ideas and the steps in the proof:

- **Step 1:** Obtain the closed loop as a switched system with a single switching signal, using the merging switching signal technique with the plant's switching signal and the controller's switching signal: $\sigma'(t) := \sigma(t) \oplus \sigma(t - \tau_s)$.
- **Step 2:** Bound the difference between x and the delayed version of x in terms of $\bar{\tau}_x$ and x using the technique in [14], [15].
- **Step 3:** Construct the candidate Lyapunov function $V(t) := V_{\sigma'(t)}(x(t))$, where V_j are as in (5), and bound V using Lemma 3 and Lemma 2 on the average dwell-time signal switching σ' . The technique in this step is similar to the technique in the original average dwell-time paper for all stable subsystems [7], which is also the technique in the extension [22] for switched systems with mixed stable and unstable subsystems. The result in [22] is not directly applicable here because although the Lyapunov functions and the technique are the same, the condition for the total time for stable and unstable modes in [22] does not match the condition for the total matched and the unmatched time for switched feedback systems with delays (see Remark 1).
- **Step 4:** Bound the Lyapunov function in Step 3 in terms of the initial state and the current state, utilizing the condition (8).
- **Step 5:** Bound the state using a small-gain technique.

Step 1: We rewrite the closed loop system as

$$\begin{aligned} \dot{x}(t) &= \bar{A}_{\sigma'(t)}x(t) + v(t), \\ v(t) &:= \bar{B}_{\sigma'(t)}(x(t - \tau_x) - x(t)), \end{aligned} \quad (20)$$

where $\bar{A}_{p,q} = A_p + B_p K_q$, $\bar{B}_{p,q} = B_p K_q$, and $\sigma'(t) := \sigma(t) \oplus \sigma(t - \tau_s)$. Let $\sigma_c(t) := \sigma(t - \tau_s)$.

From Lemma 2, we have $\sigma_c \in \mathcal{S}_{\text{ave}}[\tau_a, N_0 + \tau_s/\tau_a]$ (note that $\tau_s = \bar{\tau}_s$ in Lemma 2 because we assume constant delays here). From Lemma 1, it follows that $\sigma' \in \mathcal{S}_{\text{ave}}[\bar{\tau}_a, \bar{N}_0]$, where $\bar{\tau}_a$ and \bar{N}_0 are defined as in (7).

Step 2: For nonswitched systems, the following technique is used in [14] to deal with state delay: $x(t) - x(t - \tau_x) = \int_{t-\tau_x}^t \dot{x}(s) ds$, $t \geq t_0 + \tau_x$. For switched systems, the situation is more delicate because we can have switching in the interval $[t - \tau_x, t)$. Let $t_{s_1}, \dots, t_{s_{N(t)}}$ be the switching times in $(t - \tau_x, t)$, where $N(t)$ is the number of switches in $(t - \tau_x, t)$; by convention, $t_{s_0} = t - \tau_x$, and $t_{s_{N(t)+1}} = t$. Because no

switching occurs in $(t_{s_k}, t_{s_{k+1}})$, we have

$$\begin{aligned} &|x(t_{s_{k+1}}) - x(t_{s_k})| \\ &\leq \int_{t_{s_k}}^{t_{s_{k+1}}} |\dot{x}(s)| ds \leq (t_{s_{k+1}} - t_{s_k}) \|\dot{x}\|_{[t_{s_k}, t_{s_{k+1}}]} \\ &\leq (t_{s_{k+1}} - t_{s_k}) c_1 \|x\|_{[t_{s_k} - \tau_x, t_{s_{k+1}}]} \\ &\leq (t_{s_{k+1}} - t_{s_k}) c_1 \|x\|_{[t - 2\tau_x, t]}, \end{aligned} \quad (21)$$

where $c_1 = \sup_{p \in \mathcal{P}} \|A_p\| + \sup_{p,q} \|B_p K_q\| = c_A + c_B$. We then have

$$\begin{aligned} |x(t) - x(t - \tau_x)| &\leq \sum_{k=0}^{N(t)} |x(t_{s_{k+1}}) - x(t_{s_k})| \\ &\leq \sum_{k=0}^{N(t)} (t_{s_{k+1}} - t_{s_k}) c_1 \|x\|_{[t - 2\tau_x, t]} \\ &= \tau_x c_1 \|x\|_{[t - 2\tau_x, t]} \quad \forall t \geq t_0 + \tau_x. \end{aligned} \quad (22)$$

The foregoing inequality holds for $t \geq t_0 + \tau_x$, where t_0 is the initial time, because the integral in (21) involves the value of x in the interval $[t - 2\tau_x, t]$, and we assume that the initial state x of the closed loop is available in $[-\tau_x, 0]$ only. From the definition of v as in (20), we have

$$|v(t)| \leq \tau_x c_1 c_B |x_d(t)| \quad (23)$$

in view of $\|x\|_{[t - 2\tau_x, t]} = |x_d(t)|$.

Step 3: Let $V(t) := V_{\sigma'(t)}(x(t))$, where V_j are as in (5). Let $\bar{t}_0 := t_0 + \tau_x$. Let $T \geq \bar{t}_0$ be an arbitrary time. Denote by τ_1, \dots, τ_N the switching times of σ' in (\bar{t}_0, T) , where N is the number of switches in (\bar{t}_0, T) ; $\tau_0 = \bar{t}_0$ and $\tau_{N+1} = T^-$ by convention. Define

$$\lambda_{\sigma'(t)} := \begin{cases} -\lambda_s & \text{if } \sigma'(t) \in \mathcal{P}_s^2, \\ \lambda_u & \text{else.} \end{cases} \quad (24)$$

Because $\bar{A}_{\sigma'(t)}$ and $\bar{B}_{\sigma'(t)}$ are constant for all $t \in [\tau_k, \tau_{k+1})$, from (5) and (20) we have

$$\begin{aligned} V(t) &\leq e^{\lambda_{\sigma'(\tau_k)}(t - \tau_k)} V(\tau_k) \\ &\quad + \gamma(\tau_x c_1 c_B)^2 \int_{\tau_k}^t e^{\lambda_{\sigma'(\tau_k)}(t - s)} |x_d(s)|^2 ds \quad \forall t \in [\tau_k, \tau_{k+1}) \end{aligned} \quad (25)$$

in view of $|v(s)|^2 \leq (\tau_x c_1 c_B)^2 |x_d(s)|^2$ (from (23)). From (6b), we also have

$$V(\tau_{k+1}) \leq \mu V(\tau_{k+1}^-). \quad (26)$$

Iterating the inequalities (25) and (26) for $k = 0$ to N , we obtain

$$\begin{aligned} V(T^-) &\leq \mu^N e|_1^N(\bar{t}_0) V(\bar{t}_0) \\ &\quad + \gamma(\tau_x c_1 c_B)^2 \sum_{k=0}^N \int_{\tau_k}^{\tau_{k+1}} \mu^{N-k} e|_{k+1}^N(s) |x_d(s)|^2 ds, \end{aligned} \quad (27)$$

where

$$e|_k^N(s) := \exp\left(\sum_{i=k}^N \lambda_{\sigma'(\tau_i)}(\tau_{i+1} - \tau_i)\right) \exp(\lambda_{\sigma'(\tau_k)}(\tau_k - s)). \quad (28)$$

Step 4: The condition (8) implies the existence of λ such that

$$(\tau_x c_1 c_B)^2 \kappa + \frac{\ln \mu}{\bar{\tau}_a} < \lambda < \lambda_s - \frac{\tau_s}{\bar{\tau}_a}(\lambda_s + \lambda_u).$$

These inequalities can be written as

$$\bar{\tau}_a > \frac{\ln \mu}{\lambda} \quad (29a)$$

$$\tau_s < \frac{\lambda_s - \lambda}{\lambda_s + \lambda_u} \tau_a \quad (29b)$$

$$(\tau_x c_1 c_B)^2 \kappa < \lambda - \frac{\ln \mu}{\bar{\tau}_a}. \quad (29c)$$

From the condition (29a), we have $\lambda_s > (\ln \mu)/\bar{\tau}_a$. The conditions (29b) and (29c) can be rewritten as

$$\tau_s(\lambda_s + \lambda_u) < (\lambda_s - \lambda)\tau_a \quad (30a)$$

$$(\tau_x c_1 c_B)^2 \frac{\mu^{\bar{N}_0} \exp((\lambda_s + \lambda_u)N_0\tau_s)\alpha_2\gamma}{\alpha_1(\lambda - \ln \mu/\bar{\tau}_a)} < 1 \quad (30b)$$

in view of $\kappa := \mu^{\bar{N}_0} \exp((\lambda_s + \lambda_u)N_0\tau_s)(\alpha_2\gamma/\alpha_1)$.

We now bound the term $\mu^{N-k} e|_{k+1}^N(s)$ with exponentially decaying functions. Because $\sigma' \in \mathcal{S}_{\text{ave}}[\bar{\tau}_a, \bar{N}_0]$, and there are $N - k$ switches in $[\tau_{k+1}, T]$, we have $N - k \leq \bar{N}_0 + (T - \tau_{k+1})/\bar{\tau}_a$. Therefore, $\mu^{N-k} \leq \mu^{\bar{N}_0} \exp((T - \tau_{k+1})(\ln \mu)/\bar{\tau}_a)$. From the definition of $e|_a^b(\tau_a)$ as in (28) and the definition of $\lambda_{\sigma'}$ as in (24), aggregating the terms involving λ_s and the terms involving λ_u , we have

$$e|_a^b(\tau_a) = \exp(-\lambda_s m_{\tau_a, \tau_{b+1}} + \lambda_u \bar{m}_{\tau_a, \tau_{b+1}}). \quad (31)$$

From (30a) and Lemma 3, we have

$$-\lambda_s m_{\tau_a, \tau_{b+1}} + \lambda_u \bar{m}_{\tau_a, \tau_{b+1}} \leq c_T - \lambda(\tau_{b+1} - \tau_a), \quad (32)$$

where $c_T = (\lambda_s + \lambda_u)N_0\tau_s$. From (28), (31), and (32), we get

$$e|_a^b(\tau_a) \leq e^{c_T} \exp(-\lambda(\tau_{b+1} - \tau_a)). \quad (33)$$

We then have

$$\mu^{N-k} e|_{k+1}^N(s) \leq \mu^{\bar{N}_0} e^{c_T} e^{-\lambda'(\tau_{N+1}-s)}, \quad (34)$$

where $\lambda' := \lambda - (\ln \mu)/\bar{\tau}_a$; $\lambda' > 0$ because $\lambda \in (\ln \mu/\bar{\tau}_a, \lambda_s)$. From (27) and (34), we obtain

$$|V(T^-)| \leq \mu^{\bar{N}_0} e^{c_T} e^{-\lambda'(T-\bar{t}_0)} |V(\bar{t}_0)| \quad (35)$$

$$+ \gamma(\tau_x c_1 c_B)^2 \mu^{\bar{N}_0} e^{c_T} \int_{\bar{t}_0}^T e^{-\lambda'(T-s)} |x_d(s)|^2 ds, \quad (36)$$

$$\leq \mu^{\bar{N}_0} e^{c_T} e^{-\lambda'(T-\bar{t}_0)} \alpha_2 |x(\bar{t}_0)|^2 \quad (37)$$

$$+ (\mu^{\bar{N}_0} e^{c_T} \gamma(\tau_x c_1 c_B)^2 / \lambda') \|x_d\|_{[\bar{t}_0, T]}^2 \quad (37)$$

for all $T \geq \bar{t}_0$.

Step 5: From (35) and (6a), in view of the fact that x is continuous, we have

$$|x(T)|^2 \leq g_0 e^{-\lambda'(T-\bar{t}_0)} |x(\bar{t}_0)|^2 + (g_0 \gamma(\tau_x c_1 c_B)^2 / \lambda') \|x_d\|_{[\bar{t}_0, T]}^2 \quad \forall T \geq \bar{t}_0, \quad (38)$$

where $g_0 := \mu^{\bar{N}_0} e^{c_T} \alpha_2 / \alpha_1$. Because (38) is true for all T , we have

$$\|x_d\|_{[\bar{t}_0, T]}^2 \leq g_0 |x(\bar{t}_0)|^2 + (g_0 \gamma(\tau_x c_1 c_B)^2 / \lambda') \|x_d\|_{[\bar{t}_0, T]}^2.$$

In view of $\|x\|_{[t_0-\tau_x, t_0+\tau_x]} = |x_d(\bar{t}_0)|$ and $|x(\bar{t}_0)| \leq |x_d(t_0)|$, we have

$$\|x_d\|_{[\bar{t}_0, T]}^2 \leq \|x\|_{[t_0-\tau_x, T]}^2 \leq (1 + g_0) |x_d(\bar{t}_0)|^2 + (g_0 \gamma(\tau_x c_1 c_B)^2 / \lambda') \|x_d\|_{[\bar{t}_0, T]}^2. \quad (39)$$

From (39), given (30b), we get

$$\|x_d\|_{[\bar{t}_0, T]}^2 \leq (1 + g_0) c_2(\tau_x) |x_d(\bar{t}_0)|^2, \quad (40)$$

where $c_2(\tau_x) := (1 - g_0 \gamma(\tau_x c_1 c_B)^2 / \lambda')^{-1} > 0$ (the fact that $c_2 > 0$ follows from (30b)). From (40) and (38), we have

$$|x(T)|^2 \leq \left(g_0 e^{-\lambda'(T-\bar{t}_0)} + g_1(\tau_x)\right) |x_d(\bar{t}_0)|^2, \quad (41)$$

where $g_1(\tau_x) := (g_0 \gamma(\tau_x c_1 c_B)^2 / \lambda')(1 + g_0) c_2(\tau_x)$, and hence, in view of $e^{-\lambda'((T-2\tau_x)-\bar{t}_0)} = e^{2\lambda' \tau_x} e^{-\lambda'(T-\bar{t}_0)}$, we have

$$|x_d(T)|^2 \leq \left(g_0 e^{2\lambda' \tau_x} e^{-\lambda'(T-t_0)} + g_1(\tau_x)\right) |x_d(\bar{t}_0)|^2 \quad (42)$$

for all $T \geq \bar{t}_0$. The inequality (9) in the theorem follows from (42) by replacing \bar{t}_0 and $T \geq \bar{t}_0$ with t_0 and $t \geq t_0 + \tau_x$, respectively. The fact that $g_1(\tau_x) \rightarrow 0$ as $\tau_x \rightarrow 0$ is obvious from the definition of g_1 . ■

Remark 2 For bounded time-varying delays, i.e. τ_x and τ_s are functions of time, we still have the result claimed in Theorem 1, by replacing τ_x with $\bar{\tau}_x$ in the statement of the theorem, where $\|\tau_x\|_\infty \leq \bar{\tau}_x$, and τ_s by $\bar{\tau}_s$, where $\|\tau_s\|_\infty \leq \bar{\tau}_s$. This fact is obtained by examining the proof of the theorem, replacing τ_s by $\bar{\tau}_s$ and τ_x by $\bar{\tau}_x$ everywhere in the proof (Lemma 2 and Lemma 3 are already for time-varying delays).

Remark 3 The result in this section remains the same if (4), (5), and (6) hold for all x belonging to an invariant set Ω for all time for some set $\Omega \subseteq \mathbb{R}^n$, instead of requiring (4), (5), and (6) hold for all $x \in \mathbb{R}^n$.

The condition (8) characterizes the relationship among τ_x , τ_s , and τ_a to guarantee closed loop stability. For a fixed τ_a and N_0 , a larger τ_x requires a smaller τ_s and vice versa. When $\tau_x = 0$, τ_s is maximum and is equal to $\tau_s^* := (\tau_a \lambda_s - 2 \ln \mu) / (\lambda_s + \lambda_u)$. For a fixed τ_a and N_0 , the pairs of τ_x and τ_s satisfying (8) form a shape as the shaded area in Fig. 5a. For a fixed τ_x , a larger τ_a allows for larger switching delays τ_s and vice versa, and when $\tau_a \rightarrow \infty$, $\tau_s \rightarrow \tau_s^*$, where τ_s^* is the solution of the equation $(\tau_x c_1 c_B)^2 \mu^{2N_0} \exp((\lambda_s + \lambda_u)N_0\tau_s^*)(\alpha_2\gamma/\alpha_1) = \lambda_s$. The relationship between τ_s and τ_a for a fixed τ_x is illustrated

in Fig. 5b. For a fixed τ_s , a larger τ_a also allows for larger state delays τ_x and vice versa, and when $\tau_a \rightarrow \infty$, $\tau_x \rightarrow \tau_x^*$, where $\tau_x^* = ((\lambda_s - (\tau_s/\tau_a)(\lambda_s + \lambda_u) - 2 \ln \mu / \tau_a) / \kappa)^{1/2} / (c_1 c_B)$. The relationship between τ_a and τ_x for a fixed τ_s is illustrated in Fig. 5c.

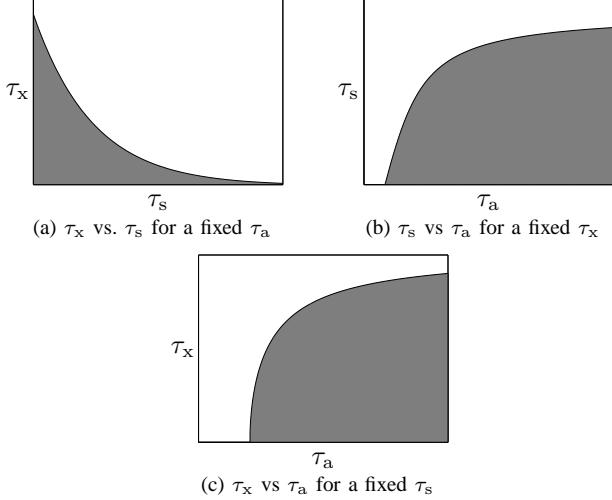


Fig. 5. Relationships among τ_s , τ_x , and τ_a . The shaded areas are the sets of feasible τ_s , τ_x , and τ_a that satisfy the condition (8) when one of those τ_s , τ_x , and τ_a is fixed.

C. Special cases

1) *Switching delay and no state delay:* The following corollary is a direct consequence of Theorem 1 when $\tau_x = 0$ (note that for $\tau_x = 0$, we have $x_d = |x|$).

Corollary 1 Consider the switched system (1) with the controller (3) in which $\tau_x = 0$. Suppose that Assumption 1 holds and that $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$ for some positive constants τ_a and N_0 . If

$$2 \frac{\ln \mu}{\tau_a} < \lambda_s - \frac{\tau_s}{\tau_a} (\lambda_s + \lambda_u), \quad (43)$$

then we have

$$|x_d(t)|^2 \leq g_0 e^{-\lambda'(t-t_0)} |x_d(t_0)|^2 \quad \forall t \geq t_0 \geq 0 \quad (44)$$

for some constants $\{\lambda', g_0\} > 0$.

2) *State delay and no switching delay:* For the special case $\tau_s = 0$, with a little tweak in the proof of Theorem 1, we have the following stronger result.

Theorem 2 Consider the switched system (1) with the controller (3) in which $\tau_s = 0$. Suppose that Assumption 1 holds and that $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$ for some positive constants τ_a and N_0 . If

$$(\tau_x c_1 c_B)^2 \kappa + \frac{\ln \mu}{\tau_a} < \lambda_s, \quad (45)$$

where $\kappa := \mu^{N_0} \alpha_2 \gamma / \alpha_1$, then we have

$$|x_d(t)|^2 \leq (g_0 e^{2\lambda' \tau_x} e^{-\lambda'(t-t_0)} + g_1(\tau_x)) |x_d(t_0)|^2 \quad (46)$$

for all $t \geq t_0 \geq \tau_x$ for some $\{g_0, \lambda'\} > 0$ and a function $g_1 : [0, \infty) \rightarrow [0, \infty)$ such that $g_1(\tau_x) \rightarrow 0$ as $\tau_x \rightarrow 0$.

Proof: In the case of no switching delay, $\sigma_c = \sigma$, and there is no mismatch between the plant's switching signal and the controller switching signals. The condition (29c) is automatically true because $\tau_s = 0$. In Step 1 in the proof of Theorem 1, we have $\sigma' = \sigma$, and so $\sigma' \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$. Then, $\bar{\tau}_a$ and \bar{N}_0 in the statement of Theorem 1 are replaced by τ_a and N_0 . From here, the proof of Theorem 2 is exactly the same as the proof of Theorem 1. ■

Theorem 2 shows that with no switching delay, asymptotic stability of feedback switched linear systems with average dwell-time switching with the same condition as in [7] (which is $\tau_a > \ln \mu / \lambda_s$) is robust to small feedback state delays.

D. Robustness of feedback switched systems with respect to delays

An important implication of Theorem 1 is that stability of feedback switched linear systems is robust with respect to state and switching delays for average dwell-time switching signals with large enough average dwell-time. We have the following result.

Corollary 2 Consider the switched system (1) with the controller (3). Suppose that Assumption 1 holds and that $\sigma \in \mathcal{S}_{\text{ave}}[\tau_a, N_0]$ for some positive constants τ_a and N_0 . If

$$\tau_a > 2 \frac{\ln \mu}{\lambda_s}, \quad (47)$$

then there exist positive numbers $\bar{\tau}_x$ and $\bar{\tau}_s$ such that the feedback switched linear system is asymptotically stable for all $\|\tau_x\|_\infty < \bar{\tau}_x$ and for all $\|\tau_s\|_\infty < \bar{\tau}_s$.

Proof: Because $(\tau_x c_1 c_B)^2 \kappa \rightarrow 0$ as $\tau_x \rightarrow 0$, and $\tau_s (\lambda_s + \lambda_u) / \tau_a \rightarrow 0$ as $\tau_s \rightarrow 0$, in view of $\bar{\tau}_a = \tau_a / 2$, the inequality (47) implies the existence of τ_x and τ_s such that

$$(\tau_x c_1 c_B)^2 \kappa + \frac{\ln \mu}{\bar{\tau}_a} < \lambda_s - \frac{\tau_s}{\tau_a} (\lambda_s + \lambda_u). \quad (48)$$

It follows from Theorem 1 that (9) holds. Because $g_1(\tau_x) \rightarrow 0$ as $\tau_x \rightarrow 0$, there exists τ_x^* such that $g_1(\tau_x^*) < 1$. Let $\bar{\tau}_x = \min\{\tau_x, \tau_x^*\}$. Clearly, there exists $T > 0$ such that

$$g_0 e^{2\lambda' \bar{\tau}_x} e^{-\lambda' T} + g_1(\bar{\tau}_x) =: g_2 < 1.$$

Then $|x_d(T + t_0)|^2 \leq g_2 |x_d(t_0)|^2$. Similarly, we have $|x_d(2T + t_0)|^2 \leq g_2 |x_d(T + t_0)|^2 \leq g_2^2 |x_d(t_0)|^2$. Repeating this argument, we have $|x_d(kT + t_0)|^2 \leq g_2^k |x_d(t_0)|^2$ for all k . Because $g_2 < 1$, we have

$$|x_d(kT + t_0)| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (49)$$

In view of (9), we have

$$\begin{aligned} |x_d(kT + t_0 + s)|^2 &\leq (g_0 e^{2\lambda' \bar{\tau}_x} + g_1(\bar{\tau}_x)) |x_d(kT + t_0)|^2 \\ &=: c |x_d(kT + t_0)|^2 \\ &\leq c g_2^k |x_d(t_0)|^2 \end{aligned} \quad (50)$$

for all $s \in [0, T)$ and for all $k = 1, \dots$. The inequality (50) implies that the system is Lyapunov stable, and that $|x_d(t)| \rightarrow 0$ as $t \rightarrow \infty$. Because $|x(t)| \leq |x_d(t)|$, we have that the system is asymptotically stable. ■

Remark 4 *In the case without state and switching delays, a feedback switched system is asymptotically stable if the switching signal has an average dwell-time of at least $\ln \mu / \lambda_s$ (see [7]). To establish robust stability with respect to delays in both the switching signal and the feedback state, we require the plant's switching signal to be two time slower than those given in [7] (i.e. $\tau_a > 2 \ln \mu / \lambda_s$). The larger lower bound of the average dwell-time in the case with delays can be thought of as a compensation for instability caused by mismatch between the switched controller and the switched plant. Whether a tighter bound on the average dwell-time can be obtained for the case with both switching and state delays remains the topic of future research. Nonetheless, the stability result reported in Theorem 1 is new for switched systems involving both switching delays and state delays.*

E. Robustness with respect to delays and unmodeled dynamics

The Lyapunov function approach also enables us address unmodeled dynamics quite straightforward. Consider a feedback switched system with unmodeled dynamics,

$$\dot{x} = A_\sigma x + B_\sigma u + \Delta(x), \quad (51)$$

in which the unmodeled dynamics satisfies

$$\|\Delta(x)\| \leq \delta \|x\| \quad \forall x \quad (52)$$

for some $\delta > 0$.

Theorem 3 *Consider the switched system (51) with the controller (3). Suppose that Assumption 1 holds, $\sigma \in \mathcal{S}_{ave}[\tau_a, N_0]$ for some positive constants τ_a and N_0 , and (52) holds for some $\delta \geq 0$. Let*

$$\begin{aligned} \bar{\tau}_a &:= \frac{\tau_a}{2} \\ \bar{N}_0 &:= 2N_0 + \frac{\tau_s}{\tau_a}. \end{aligned} \quad (53a)$$

If

$$(\tau_x c_1 c_B + \delta)^2 \kappa + \frac{\ln \mu}{\bar{\tau}_a} < \lambda_s - \frac{\tau_s}{\tau_a} (\lambda_s + \lambda_u), \quad (54)$$

where $\kappa := \mu^{\bar{N}_0} \exp((\lambda_s + \lambda_u) N_0 \tau_s) (\alpha_2 \gamma / \alpha_1)$, then we have

$$|x_d(t)|^2 \leq \left(g_0 e^{2\lambda' \tau_x} e^{-\lambda'(t-t_0)} + g_1(\tau_x) \right) |x_d(t_0)|^2 \quad (55)$$

for all $t \geq t_0 \geq \tau_x$ for some constants $\{\lambda', g_0\} > 0$ and function $g_1 : [0, \infty) \rightarrow [0, \infty)$ such that $g_1(\tau_x) \rightarrow 0$ as $\tau_x \rightarrow 0$.

Proof: The proof is almost the same as that of Theorem 1 with a few alterations. In (20), the variable v is changed to $\bar{B}_{\sigma'(t)}(x(t - \tau_x) - x(t)) + \Delta(x(t))$. The inequality (23) is changed to

$$|v(t)| \leq (\tau_x c_1 c_B + \delta) |x_d(t)| \quad (56)$$

in view of $|\Delta(x(t))| \leq \delta |x(t)| \leq \delta |x_d(t)|$. Everything else is the same as in the proof of Theorem 1 with $\tau_x c_1 c_B$ in the proof of Theorem 1 replaced by $\tau_x c_1 c_B + \delta$ everywhere after (23). ■

In view of Corollary 2, Theorem 3 also implies that stability of feedback switched systems is robust to state delays, switching delays, and linearly bounded unmodeled dynamics when all of these quantities are present.

V. APPLICATION: CONSENSUS NETWORKS WITH SWITCHING TOPOLOGY AND TIME-VARYING DELAY

Consider a network of n agents with interactions dictated by an undirected graph (topology), whose agents' dynamics are $\dot{x}_i = u_i, i = 1, \dots, n$, and without loss of generality, assume that $x_i \in \mathbb{R}$. Suppose that each agent employs the consensus protocol $u_i = \sum_{j \in \mathcal{N}_i} x_j - x_i$, where \mathcal{N}_i is the neighborhood of agent i . The reader is referred to [23], for example, for background on graph theory. Further, suppose that the network has a uniform one-hop time-varying communication delay $\tau_x(t)$ among the agents, so that the actual control signal is

$$u_i(t) = \sum_{j \in \mathcal{N}_i} x_j(t - \tau_x(t)) - x_i(t) \quad (57)$$

(there is no state delay for information from the same agent). The work [24] considers $u_i(t) = \sum_{j \in \mathcal{N}_i} x_j(t - \tau_x) - x_i(t - \tau_x)$ and without topology switching, under which the collective dynamics are a non-switched linear system with delays. The authors in [24] then obtained a necessary and sufficient condition on the allowable delay for stability, utilizing well-known techniques (Nyquist criterion) for delay linear systems.

With the protocol (57), the collective dynamics will be of the form of a switched system with delays (as discussed in the motivation section, Section II-B). The tool we developed in the previous section helps us establish the following result. Recall that a graph is k -regular if every node has degree k . We say that a network of agents asymptotically reaches consensus if $x_i(t) - x_j(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $i \neq j, i, j \in \{1, \dots, n\}$.

Theorem 4 *Consider a multi-agent network with the protocol (57) and a switching topology $G : [0, \infty) \rightarrow \{G_1, \dots, G_m\}$, where G_p are k_p -regular undirected graphs for some number $k_p, p = 1, \dots, m$. For every $\tau_a > 0$ and $N_0 > 0$, there exists a number $\bar{\tau}_x > 0$ such that if the switching signal of the switching topology belongs to $\mathcal{S}_{ave}[\tau_a, N_0]$ and the delay $\tau_x(t) < \bar{\tau}_x \forall t$, then for all the initial states, the network of agents will asymptotically reach consensus.*

Proof: Let $x = (x_1, \dots, x_n)^T$ be the collective state of the network. Define

$$A_p := A_{G_p}, \quad D_p := k_p I \quad (58)$$

where A_p is the adjacency matrix of G_p , and D_p is the degree matrix of G_p (recall that G_p is k_p -regular). Also, define the switching signal $\sigma : [0, \infty) \rightarrow \{1, \dots, m\}$ such that

$$\sigma(t) := p \text{ where } G(t) = G_p, \quad p \in \{1, \dots, m\}, \quad \forall t \geq 0. \quad (59)$$

By direct inspection, the dynamics of the network are

$$\dot{x}(t) = -D_{\sigma(t)}x(t) + A_{\sigma(t)}x(t - \tau_x).$$

Let $\delta(t) := x(t) - \frac{1}{n}(\mathbf{1}^T x(t))\mathbf{1} =: F(x(t))$, where $\mathbf{1} \in \mathbb{R}^n$ is vector whose elements are all ones (the vector δ is known as the *disagreement vector* [24].) Then $\mathbf{1}^T \delta = 0$. We have

$$\begin{aligned} \dot{\delta} &= \dot{x} - (1/n)(\mathbf{1}^T \dot{x})\mathbf{1} = F(\dot{x}) \\ &= F(-D_{\sigma(t)}x(t) + A_{\sigma(t)}x(t - \tau_x)) \end{aligned}$$

in view of the fact that $F(x)$ is linear in x over real numbers.

For a k_p -regular graph G_p , we have $\mathbf{1}^T A_p = k_p \mathbf{1}^T$ and $A_p \mathbf{1} = k_p \mathbf{1}$, and so

$$(\mathbf{1}^T A_p x)\mathbf{1} = k_p (\mathbf{1}^T x)\mathbf{1} = (\mathbf{1}^T x)k_p \mathbf{1} = (\mathbf{1}^T x)A_p \mathbf{1} = A_p (\mathbf{1}^T x)\mathbf{1}.$$

The preceding equality implies that $F(A_{\sigma(t)}x(t - \tau_x)) = A_{\sigma(t)}F(x(t - \tau_x))$. Then

$$\begin{aligned} \dot{\delta}(t) &= -F(k_{\sigma(t)}Ix(t)) + F(A_{\sigma(t)}x(t - \tau_x)) \\ &= -k_{\sigma(t)}\delta(t) + A_{\sigma(t)}F(x(t - \tau_x)) \\ &= -k_{\sigma(t)}\delta(t) + A_{\sigma(t)}\delta(t - \tau_x) \end{aligned} \quad (60)$$

where the last equality follows from the definition of δ that $\delta(t - \tau_x) = F(x(t - \tau_x))$.

The system (60) can be further cast as the following switched plant

$$\dot{\delta} = -k_{\sigma}\delta + u \quad (61)$$

with the delay feedback controller

$$u = A_{\sigma}\delta(t - \tau_x). \quad (62)$$

Now, for every $p \in \{1, \dots, m\}$, we have $A_p - k_p I = -L_p$, where L_p is the Laplacian of the graph G_p . For any undirected graph or balanced graph G , the graph Laplacian L_G has the following property (see, e.g., [24]):

$$\delta^T L_G \delta \geq \lambda_2(L_G)|\delta|^2 \quad \forall \delta : \mathbf{1}^T \delta = 0, \quad (63)$$

where $\lambda_2(L_G)$ is the smallest nonzero eigenvalue of L_G . Then the quadratic function $V = \delta^T \delta$ has the following property: along the trajectory of $\dot{\delta} = -L_p \delta + v$,

$$\begin{aligned} \dot{V} &= -2\delta^T(L_p \delta + v) \\ &\leq -(2\lambda_2(L_{G_p}) - \epsilon)V - \epsilon(|\delta| - 1/\epsilon|v|)^2 + 1/\epsilon|v|^2 \\ &\leq -(2\lambda_2(L_{G_p}) - \epsilon)V + 1/\epsilon|v|^2 \end{aligned} \quad (64)$$

for some $\epsilon \in (0, 2\lambda_2(L_{G_p}))$. The last inequality shows that V satisfies the condition (4) with $\lambda_s := \min_p 2\lambda_2(L_{G_p}) - \epsilon$ and $\gamma = 1/\epsilon$ for some $\epsilon \in (0, \min_i 2\lambda_2(L_{G_p}))$. Therefore, the set \mathcal{P}_s^2 as in (4) is nonempty (it contains at least all elements of the form (p, p) , $p = 1, \dots, m$). In this case of multi-agent networks without switching delay, the set \mathcal{P}_u^2 is empty.

Thus, the feedback switched system (61) with the switched controller (62) satisfies the condition $\mathcal{P}_s^2 \neq \emptyset$. From Corollary 2 and Remark 3 with $\Omega = \{z : \mathbf{1}^T z = 0\}$, if $\sigma \in \mathcal{S}_{ave}[\tau_a, N_0]$ and $\tau_a > \ln \mu / \lambda_s$, then $\bar{\tau}_x$ exists such that for all initial states and all delays $\tau_x < \bar{\tau}_x$, we have $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$. From the definition of δ , we have $x_i(t) \rightarrow \frac{1}{n} \sum_{j=1}^n x_j(t)$ as $t \rightarrow \infty$ for

all i , and hence, network consensus is asymptotically achieved. This result is true for arbitrarily small τ_a because $V = \delta^T \delta$ is the common Lyapunov function, so $\mu = 1$, and we always have $\ln \mu / \tau_a < \lambda_s$ for all $\tau_a > 0$. ■

For illustration, consider a network of 5 agents whose topology is switching between the two topologies in Fig. 6. Pick $\epsilon = 1$, where ϵ is as in (64). For the topology in Fig. 6, we then calculate $\lambda_s = 10.226$, $\lambda_u = 4$, $\gamma = 1$, and $\mu = 1$. Then $\bar{\tau}_x = 0.095$ would satisfy (8) when $\tau_s = 0$. The simulation result with $\tau_a = 0.01$, $N_0 = 1$, $x(0) = (1, 2, 3, 4, 5)^T$, and τ_x varying uniformly between 0 and $\bar{\tau}_x$ is shown in Fig. 7.

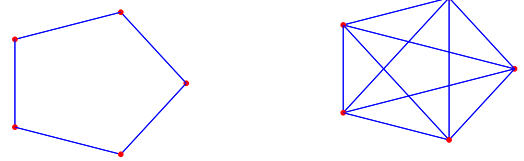


Fig. 6. Topology switching between a 2-regular topology and a 4-regular topology.

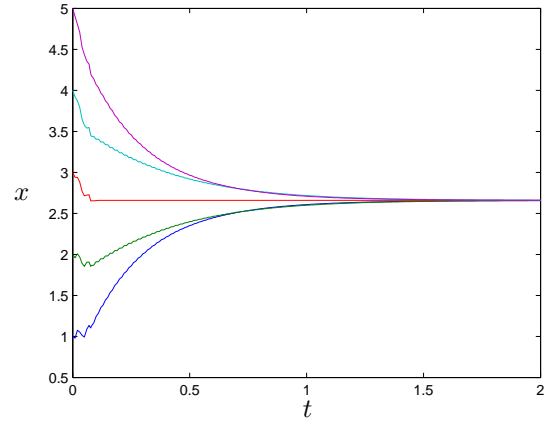


Fig. 7. Simulation result with $\tau_a = 0.01$, $N_0 = 1$, $x(0) = (1, 2, 3, 4, 5)^T$, and τ_x varying uniformly between 0 and $\bar{\tau}_x$.

VI. LMI METHODOLOGY FOR NUMERICAL CALCULATION

In this section, we provide a methodology based on LMIs for numerically obtaining bounds on the delays to guarantee closed loop stability for given feedback controllers.

In the condition (8), the left hand side involves the constant κ , which in turns depends on μ , γ , and the ratio α_2/α_1 . Thus, to get a larger bound on τ_s and τ_x , it is desirable to obtain small μ , γ , and α_2/α_1 . For given A_p, B_p , and K_p , there are many choices of $P_i, i \in \mathcal{P}^2$ as in (4) and (5), corresponding to many possible μ, γ , and α_2/α_1 as in (6). A way of finding small numerical μ, γ , and α_2/α_1 is to use linear-matrix-inequalities (LMI). We seek quadratic $V_i(x) = x^T P_i x, i \in \mathcal{P}_s^2$, where

$$P \succ 0, \quad (65)$$

such that the inequality (4) holds (the expression $P \succ 0$ denotes that P is positive definite; similarly, for two matrices

P and Q , $P \prec Q$ means that $Q - P$ is positive definite). The inequality (4), in explicit form, is

$$x^T(A_j^T P_j + P_j A_j)x + 2x^T P_j v \leq -\lambda_s x^T P_j x + \gamma v^T v,$$

which leads to the following LMI:

$$\begin{bmatrix} \overline{A}_j^T P_j + P_j \overline{A}_j + \lambda_s P_j & P_j \\ P_j & -\gamma I \end{bmatrix} \prec 0, \quad j \in \mathcal{P}_s^2 \quad (66)$$

(we replace inequality by strict inequality.) Similarly, (5) leads to

$$\begin{bmatrix} \overline{A}_j^T P_j + P_j \overline{A}_j - \lambda_u P_j & P_j \\ P_j & -\gamma I \end{bmatrix} \prec 0, \quad j \in \mathcal{P}_u^2, \quad (67)$$

and (6a) and (6b) can be written as

$$\begin{aligned} \alpha_1 I \prec P_i \prec \alpha_2 I, \quad i \in \mathcal{P}^2 \\ P_i \prec \mu P_j, \quad i, j \in \mathcal{P}^2. \end{aligned} \quad (68)$$

For given $\lambda_s, \lambda_u, \alpha_1, \alpha_2, \gamma$, and λ , the set of LMIs (65), (66), (67), and (68) can be solved numerically for the existence of P_i using computational software (such as the MATLAB LMI toolbox). One can then start from the smallest $\mu = 1$, a large γ , and a large α_2/α_1 and incrementally increase μ and incrementally decrease γ and α_2/α_1 as far as possible while the set of LMIs still has a solution.

Example 1 Consider the feedback switched system (1) with the subsystems

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (69)$$

$$A_2 = \begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (70)$$

Suppose that the switched state feedback controllers (3) are

$$K_1 = \begin{bmatrix} -7 & -6 \end{bmatrix} \quad (71)$$

$$K_2 = \begin{bmatrix} 5 & -3 \end{bmatrix}, \quad (72)$$

which places the closed-loop poles at $-1, -2$ for each of the respective closed loops. Using the LMIs (66), (67), and (68), we found that $P_i, i = 1, 2$, exist with $\lambda_s = 0.2, \lambda_u = 15, \gamma = 10^{-3}$, and $\mu = 3.8$. Suppose that $N_0 = 2$. Using the formulae $\tau_a = 2 \ln \mu / \lambda + 0.001, \lambda = \lambda_s - 0.01, \tau_s = (\lambda_s - \lambda) / (\lambda_s + \lambda_u) \tau_a, \overline{N}_0 = 2N_0 + \tau_s / \tau_a$, and $\tau_x = ((\lambda - 2 \ln \mu / \tau_a) / \mu^{\overline{N}_0} \exp((\lambda_s + \lambda_u) N_0 \tau_s) (\alpha_2 \gamma / \alpha_1))^{1/2} / (c_1 c_B)$ (which is derived from (8)), we get $\tau_a = 14.1526, \tau_s = 0.0093$, and $\tau_x = 2.2601 \times 10^{-4}$.

VII. CONCLUSIONS

In this paper, we have studied stability of feedback switched linear systems with both state and switching delays. When the switched plant's switching signal was an average dwell-time switching signal, we provided a condition in terms of an upper bound on the delays and in terms of a lower bound on the average dwell-time of the plant's switching signal to guarantee asymptotic stability. The stability result also implies that stable feedback switched linear systems are robust to

both state delays and switching delays, as well as to small linear additive unmodeled dynamics. We applied our results in switched systems to show stability of multi-agents dynamics networks with switching topology with small state delays and arbitrarily fast switching. A numerical calculation method based on LMIs and numerical examples are also provided. Future work aims to extend the results here to the output feedback case, to switched nonlinear systems (using the tools in [14]), and to other classes of slow switching signals (as in [6]).

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