

Controlled Collective Motion for Trajectory Tracking

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Abstract—This paper extends previous work on oscillator models to meet the needs of multiagent applications in which the motion of the collective centroid of the group must be dynamic. Individual agents are modeled as unit speed planar kinematic unicycles. A steering control law is derived for each individual so that the velocity of the collective centroid will match a reference velocity, provided the reference speed is less than one. A framework for steering controls is presented such that the unicycles stay near the collective centroid even though the centroid is non-static. Finally, an outer loop controller is proposed to allow tracking of a target vehicle. Simulation results are shown to support analysis.

I. INTRODUCTION

With the growing number of applications based on coordination or cooperation, dynamics and control of multiagent systems has been the topic of many recent papers in the control theoretic literature. While in many cases analysis of such systems has assumed agent dynamics that are linear single or double integrators, in reality most vehicles are more appropriately modeled as nonholonomic. Further, many approaches to multiagent systems lead to challenging mathematical formulations which often become intractable for in-depth analysis. In recent work, a combination of oscillator models, studied extensively in physics and mathematics, and planar Frenet-Serret frame dynamics, which are equivalent to the nonholonomic planar unicycle, has led to analysis which is both tractable and admits nonholonomic vehicle dynamics.

Early work on groups of unit speed planar unicycles was presented in [1], [2] where a Lie group formulation on $SE(2)$ was proposed for two vehicles with unit speed unicycle kinematics. The proposed steering control laws were based on heading alignment, relative spacing, and baseline perpendicularity. Analysis showed that the system of two vehicles was stable to equilibria sets where all vehicles had the same heading, termed the *aligned* set, or where all vehicles circled a fixed group centroid, termed the *balanced* set.

A connection between coupled oscillator theory and multi-unicycle control was established in [3]. This connection was based on singular perturbation theory, which allowed alignment and spacing control to be decoupled. When singularly perturbed, the model studied in [1], [2] is precisely the well known Kuramoto oscillator model [4]. More recent work has focused on the spacing of individual vehicles within the balanced set [5], [6]. One special case of spacing in

the balanced set is the *splay state*, which is characterized by uniform rotation of evenly spaced vehicles on a circle. Among other results, these works presented a steering control law that was asymptotically stable to the splay state. Further work in this area can be found in [3], [6], [7], [8]. Related use of oscillator models with coupled unit speed unicycles has focused on discrete time models [9], network topology [10], [11], and extensions to higher dimensions [12].

In each of the studies discussed above, the control laws stabilized the speed of the centroid to either a minimum (zero) or to a maximum (one). Therefore, these control laws cannot be directly used for applications such as trajectory tracking that require the centroid to travel at intermediate speeds. The general class of applications which require a group of constant speed vehicles to track a *moving* target has not been studied extensively in the literature although this task is of great interest in a number of scenarios. The single vehicle problem has been explored in the aeronautics community [13], [14], [15]; however, for multiple vehicles, the closest example is a group of unit speed unicycles capable of switching between circling and aligned behavior primitives [6]. To address the needs of applications with moving targets, results are presented here which encompass the prior work and, for the general class of systems with multiple unit speed agents, extend to the setting where the agents are required to track a target vehicle, real or virtual, which has a nonconstant velocity.

The paper is organized as follows. In Section II, the system dynamics are presented and our general approach is outlined. In Section III, a control law is derived and shown to stabilize the collective centroid to any constant reference velocity, provided the speed of the reference is less than one. These results are then refined using a feedforward control term to stabilize the collective centroid to a nonconstant reference velocity. The case of a variable speed unicycle target vehicle is used as an example. A framework for making individual vehicles stay near the collective centroid using spacing control is presented in Section IV, and a specific control law for $N = 3$ is given. Finally, an outer loop controller is developed in Section V and used to compensate for errors in the feedforward control term and to regulate spatial error between the target and the collective centroid. Throughout the work, simulations of a group of unmanned aerial vehicles observing a target tank on the ground will demonstrate the methods. Directions for future work are given in Section VI.

This work was supported in part by NSF grant CCR-0313250 and in part by AFOSR grants F054-011-0220 and FA9550-05-C-0112.

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II. PROBLEM FORMULATION

The primary task to be considered here is the tracking of a target vehicle's position, r_t , by the centroid, \bar{r} , of a homogeneous group of N unit speed unicycles with dynamics

$$\frac{d}{dt} \begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} e^{i\theta_k} \\ u_k \end{bmatrix} \quad k = 1 \dots N. \quad (1)$$

Here $r_k \in \mathbb{C}$ and $\theta_k \in \mathbb{S}^1$ specify the $SE(2)$ position and orientation of the k^{th} unicycle, and u_k is the steering control input. Further, the individual unicycles are desired to stay near the centroid (and hence the target vehicle). All-to-all continuous-time communication and perfect sensing are assumed for each vehicle and inter-vehicle collisions are ignored.

The approach here utilizes a *reference velocity*, which is the commanded velocity of the centroid of the group. Due to the unit speed constraint of each unicycle, the maximum possible speed of the centroid is one, and the reference velocity is therefore limited to have norm less than one. Stability of the error dynamics between the centroid velocity and the reference velocity are treated separately from automatic generation of the reference velocity. Starting with the simple case of a constant reference velocity, a steering control law is derived which asymptotically stabilizes the velocity of the collective centroid to the reference velocity. These results are then extended to allow a dynamic reference velocity. This portion of the steering controller is termed *matching control*,

$$u_k = u_k^{match}, \quad (2)$$

because it causes the velocity of the centroid to match the reference velocity.

To keep individual unicycles near the centroid while the centroid velocity matches the reference velocity, a *spacing control* term is added to the steering controller on each individual,

$$u_k = u_k^{match} + u_k^{space}. \quad (3)$$

This additional spacing control term is shown to not interfere with velocity matching.

Finally, automatic generation of the reference velocity is considered in order to accomplish the overall goal of tracking a target vehicle. The target vehicle need not be a physical entity, but all group members are assumed to have knowledge of it as if it were physically present. This part of the control can be seen as an outer loop to the velocity matching theory, and unlike the previous parts, the physical location of the target vehicle (and hence the group) is considered explicitly.

III. MATCHING A REFERENCE VELOCITY

The full problem of tracking a target vehicle will be broken into components starting with the subproblem of regulating the error between the velocity of the collective centroid, $\dot{\bar{r}}$, and a reference velocity, \dot{r}_{ref} ,

$$e_v \equiv \dot{\bar{r}} - \dot{r}_{ref}. \quad (4)$$

Note that the physical locations of each unicycle and the group centroid are not treated here as the only concern is

velocity matching. The goal is to determine a steering control law of the form $\dot{\theta}_k = u_k^{match}$ for $k = 1, \dots, N$. To find this control law, one need only consider the heading of each vehicle as the velocity of the group centroid is a function of headings only:

$$\dot{\bar{r}}(\boldsymbol{\theta}) = \frac{1}{N} \sum_k e^{i\theta_k} \quad (5)$$

where $\boldsymbol{\theta} = [\theta_1 \dots \theta_N]^T$. For this reason one can often work with the heading-only subsystem which consists of the reference velocity and the headings of each unicycle.

Theorem 1: Alignment Control for Matching a Constant Reference Vector. For a *constant* reference velocity, the steering control law,

$$u_k^{match} = -K \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, i e^{i\theta_k} \rangle, \quad K > 0, \quad (6)$$

where K is a control gain, will drive the velocity of the collective centroid to the reference velocity provided $|\dot{r}_{ref}| < 1$.

Proof: The proof follows from LaSalle's Invariance Principle. Consider the candidate Lyapunov function

$$\begin{aligned} V(\boldsymbol{\theta}) &= \frac{1}{2} N \|e_v\|^2 = \frac{1}{2} N \|\dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}\|^2 \\ &= \frac{1}{2} N \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref} \rangle. \end{aligned} \quad (7)$$

This function is everywhere strictly positive except at points where the velocity of the centroid matches the reference velocity. Below, these points are shown to be the stable equilibria of the system.

The time derivative of the Lyapunov function along trajectories of the system is

$$\dot{V}(\boldsymbol{\theta}) = N \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, \ddot{\bar{r}}(\boldsymbol{\theta}) \rangle, \quad (8)$$

because \dot{r}_{ref} is constant with respect to time. Now note that

$$\begin{aligned} \frac{\partial V}{\partial \theta_k} &= N \left\langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, \frac{\partial \dot{\bar{r}}}{\partial \theta_k} \right\rangle \\ &= N \left\langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, \frac{1}{N} i e^{i\theta_k} \right\rangle. \end{aligned} \quad (9)$$

The proposed control law (6) is a gradient controller because $u_k^{match} = -K \partial V / \partial \theta_k$. Using $\dot{\theta}_k = u_k^{match}$, the centroid acceleration is

$$\begin{aligned} \ddot{\bar{r}}(\boldsymbol{\theta}) &= \frac{1}{N} \sum_k i e^{i\theta_k} u_k^{match} \\ &= -\frac{K}{N} \sum_k i e^{i\theta_k} \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, i e^{i\theta_k} \rangle. \end{aligned} \quad (10)$$

Equations (5) and (10) can be substituted into (8) to get

$$\begin{aligned} \dot{V}(\boldsymbol{\theta}) &= N \left\langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, \right. \\ &\quad \left. -\frac{K}{N} \sum_k i e^{i\theta_k} \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, i e^{i\theta_k} \rangle \right\rangle. \end{aligned} \quad (11)$$

Because the inner product is a linear operator, the last equation becomes

$$\dot{V}(\boldsymbol{\theta}) = -K \sum_k \langle \dot{\bar{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}, i e^{i\theta_k} \rangle^2, \quad (12)$$

which is negative semi-definite for $K > 0$.

Because $\boldsymbol{\theta} \in \mathbf{T}^N$ and \mathbf{T}^N is compact, by LaSalle's Invariance Principle all solutions converge to the largest invariance set contained in

$$E = \{\boldsymbol{\theta} \mid \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}, ie^{i\theta_k} \rangle = 0 \forall k\}. \quad (13)$$

All points in this set are invariant with respect to the heading-only system because

$$u_k^{match} = -K \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}, ie^{i\theta_k} \rangle \quad (14)$$

is zero for all points in E .

The set E contains both stable and unstable equilibria. The unstable equilibria are characterized by $\langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}, ie^{i\theta_k} \rangle = 0$ and $\dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) \neq \dot{\mathbf{r}}_{ref}$. At these points, the velocity error vector is parallel to every vehicle, so the Lyapunov function (7) is non-decreasing. A small perturbation away from this singularity will allow the steering control to descend the Lyapunov function. At the remaining points, $\dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) = \dot{\mathbf{r}}_{ref}$, as desired. These points are stable because of the gradient nature of the control law. In other words, the heading-only system will converge to points in E where the velocity of the collective centroid matches the (constant) reference velocity. ■

If the reference velocity is dynamic, velocity matching can be achieved using open-loop feedforward control based on knowledge of the dynamics and control inputs of the reference velocity. For now, assume that a deterministic model of the reference velocity is available. Later, the reference velocity will be automatically generated based on a target vehicle. If the reference velocity is synthetically generated, knowledge of the dynamic model and control inputs can be exact. For applications in which the reference is based on a real vehicle, a state estimator can be used in conjunction with an approximate dynamic model. In section V, an outer loop controller is considered that can compensate for small errors in the assumed dynamic model of the reference. First the previous result is extended to dynamic reference velocities as follows.

Theorem 2: Alignment Control for Matching a Dynamic Reference Velocity. The steering control law

$$u_k^{match} = -K \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}(t), ie^{i\theta_k} \rangle + h_k(\boldsymbol{\theta}, \ddot{\mathbf{r}}_{ref}) \quad (15)$$

will drive the velocity of the collective centroid to the *dynamic* reference velocity, provided $|\dot{\mathbf{r}}_{ref}(t)| < 1$ and $\mathbf{h} \in \mathbb{R}^N$, a feedforward control term, satisfies

$$\frac{1}{N} \sum_k ie^{i\theta_k} h_k = \ddot{\mathbf{r}}_{ref}(t). \quad (16)$$

Proof: The proof follows that of Theorem 1 until calculation of the time derivative of the Lyapunov function (8). Here $\dot{\mathbf{r}}_{ref}(t)$ is not constant, so

$$\dot{V}(\boldsymbol{\theta}, t) = N \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}(t), \ddot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \ddot{\mathbf{r}}_{ref}(t) \rangle. \quad (17)$$

With the addition of \mathbf{h} to the control, the acceleration of the collective centroid becomes

$$\begin{aligned} \ddot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) &= \frac{-K}{N} \sum_k ie^{i\theta_k} \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}(t), ie^{i\theta_k} \rangle \\ &\quad + \frac{1}{N} \sum_k ie^{i\theta_k} h_k. \end{aligned} \quad (18)$$

When substituted back into (17), the term involving h_k cancels with $\ddot{\mathbf{r}}_{ref}(t)$, giving

$$\dot{V}(\boldsymbol{\theta}, t) = -K \sum_k \langle \dot{\tilde{\mathbf{r}}}(\boldsymbol{\theta}) - \dot{\mathbf{r}}_{ref}(t), ie^{i\theta_k} \rangle^2 \quad (19)$$

as in Theorem 1, and the result follows. ■

As an example of this approach, a feedforward controller for a reference velocity that has unicycle dynamics is constructed as follows.

Corollary 1: Unicycle Reference Velocity. If the reference velocity has variable speed unicycle dynamics,

$$\begin{aligned} \dot{\mathbf{r}}_{ref} &= s_{ref} e^{i\theta_{ref}} \\ \dot{\theta}_{ref} &= u_{ref} \\ \dot{s}_{ref} &= a_{ref}, \end{aligned} \quad (20)$$

then the feedforward control \mathbf{h} exists provided all vehicles do not simultaneously share parallel headings. In this case \mathbf{h} is given by

$$\mathbf{h} = V\Sigma^+U^{-1}b \quad (21)$$

where

$$b = \begin{bmatrix} a_{ref} \cos(\theta_{ref}) - s_{ref} \sin(\theta_{ref}) u_{ref} \\ a_{ref} \sin(\theta_{ref}) + s_{ref} \cos(\theta_{ref}) u_{ref} \end{bmatrix} \quad (22)$$

and U, Σ, V are the singular value decomposition of

$$A = \frac{1}{N} \begin{bmatrix} -\sin(\theta_1) & -\sin(\theta_2) & \dots & -\sin(\theta_N) \\ \cos(\theta_1) & \cos(\theta_2) & \dots & \cos(\theta_N) \end{bmatrix}. \quad (23)$$

Proof: The feedforward steering control term must satisfy

$$\frac{1}{N} \sum_k ie^{i\theta_k} h_k = \ddot{\mathbf{r}}_{ref} = a_{ref} e^{i\theta_{ref}} + s_{ref} ie^{i\theta_{ref}} u_{ref}. \quad (24)$$

Associating the velocity of each vehicle with \mathbb{R}^2 in the usual way, (24) can be rewritten as

$$A\mathbf{h} = b \quad (25)$$

where A and b are as in the Corollary statement.

For $N = 2$ this equation has a single solution

$$\mathbf{h} = A^{-1}b, \quad (26)$$

provided the vehicles are not moving parallel to one another (to ensure A is invertible). For $N \geq 3$, multiple solutions for \mathbf{h} exist. A simple choice is the solution with least norm, which can be found from a SVD-pseudoinverse of A as

$$\begin{aligned} A &= U\Sigma V^{-1} \\ \mathbf{h} &= V\Sigma^+U^{-1}b, \end{aligned} \quad (27)$$

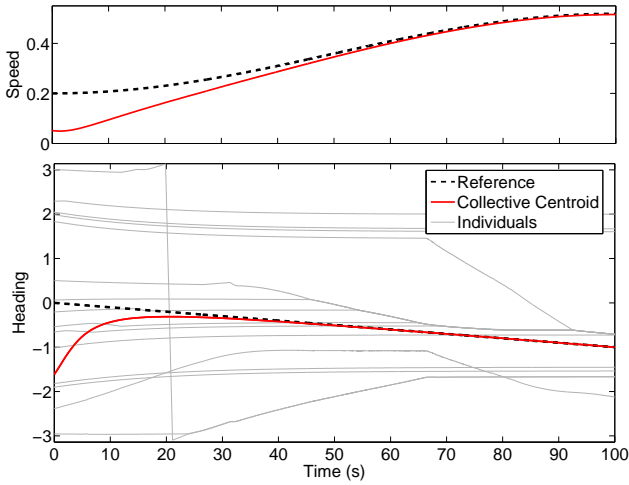


Fig. 1. The centroid of a group of $N = 15$ unit speed unicycles tracks a reference velocity that is modeled as a unicycle. The reference velocity simultaneously accelerates (top) and turns (bottom). The control gain was $K = 0.1$.

where Σ^+ has dimensions of Σ^T with all non-zero elements inverted. A solution will exist provided not all vehicles share parallel headings simultaneously. ■

A demonstration of this technique is shown in Fig. 1 with a group of fifteen vehicles whose centroid is tracking a reference velocity that was modeled as a planar unicycle. Note that only the headings of the individual vehicles are shown. The positions diverge from the centroid, a problem which is considered in the next section.

IV. STAYING NEAR THE COLLECTIVE CENTROID

Thus far, the analysis has only considered the velocity of the collective centroid; the spatial location of each vehicle has been neglected. This issue is remedied by adding a steering control term that keeps individuals near the collective centroid *while* the velocity of the collective centroid matches the reference velocity. A framework based on periodic steering control is developed here to support this objective, and an example is presented for $N = 3$ vehicles.

A single unit speed unicycle with T -periodic controls is said to *stay near* the centroid over period T when the average distance between the unicycle and the centroid is nonincreasing. Hence, the change in squared distance,

$$D_k(t, T) = \rho_k(t + T)^2 - \rho_k(t)^2, \quad (28)$$

where $\rho_k(t)$ is the distance between vehicle k and the centroid at time t , is desired to be nonincreasing for some T and all k and t . This definition allows individuals to be far from the centroid for long periods of time as long as they return in time T . Of primary interest are small- T trajectories (and hence steering controls) that satisfy this criteria. A small period ensures individuals will not stray too far from the centroid before returning, however as $T \rightarrow 0$, staying near the centroid is only feasible for $s_{ref} \rightarrow 1$.

In addition to each vehicle staying near the centroid on T -period average intervals, the controls should incorporate

collective velocity matching from the previous section. The velocity matching constraint can be written as,

$$\frac{1}{N} \sum_k e^{i\theta_k} = \dot{r}_{ref}(t) \quad (29)$$

where $|\dot{r}_{ref}(t)| < 1$. The set of states for which the constraint (29) holds is referred to as the *matched set* which can be written as

$$\mathcal{M} \equiv \left\{ \boldsymbol{\theta} \mid \frac{1}{N} \sum_k e^{i\theta_k} - \dot{r}_{ref}(t) = 0 \right\}. \quad (30)$$

Theorem 3: Combined Matching and Spacing. Any spacing control that satisfies,

$$\mathbf{u}^{space} \in \ker \{J\} \quad (31)$$

where

$$J \equiv \frac{i}{N} [e^{i\theta_1} \quad e^{i\theta_2} \quad \dots \quad e^{i\theta_N}] \quad (32)$$

will not disrupt velocity matching. If the state, $\boldsymbol{\theta}$, is in \mathcal{M} , then \mathbf{u}^{space} must be in $T\mathcal{M}(\boldsymbol{\theta})$, the tangent space to \mathcal{M} at the point $\boldsymbol{\theta}$.

Proof: Returning to the Lyapunov function given in (7), the addition of spacing control gives

$$\dot{\theta}_k = u_k = u_k^{match} + u_k^{space}. \quad (33)$$

The time derivative of (7) is now

$$\begin{aligned} \dot{V}(\boldsymbol{\theta}, t) &= N \left\langle \dot{\tilde{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}(t), \frac{1}{N} \sum_k i e^{i\theta_k} u_k - \ddot{r}_{ref}(t) \right\rangle \\ &= N \left\langle \dot{\tilde{r}}(\boldsymbol{\theta}) - \dot{r}_{ref}(t), \frac{1}{N} \sum_k i e^{i\theta_k} u_k^{match} \right. \\ &\quad \left. + \frac{1}{N} \sum_k i e^{i\theta_k} u_k^{space} - \ddot{r}_{ref}(t) \right\rangle. \end{aligned} \quad (34)$$

The term involving the spacing control drops out because

$$\frac{1}{N} \sum_k i e^{i\theta_k} u_k^{space} = J \mathbf{u}^{space} = 0. \quad (35)$$

The remaining terms in \dot{V} are exactly as considered in Theorem 2. Therefore, any spacing control in the kernel of J is admissible because it will not disturb velocity matching.

To see the relationship of \mathbf{u}^{space} with the tangent space to \mathcal{M} at $\boldsymbol{\theta}$, recall that the tangent space can be computed as the kernel of the Jacobian matrix. Using the definition of \mathcal{M} from (30),

$$T\mathcal{M}(\boldsymbol{\theta}) \equiv \ker \left\{ \frac{i}{N} [e^{i\theta_1} \quad e^{i\theta_2} \quad \dots \quad e^{i\theta_N}] \mid \boldsymbol{\theta} \in \mathcal{M} \right\}, \quad (36)$$

clearly J is the Jacobian matrix of \mathcal{M} so that $\mathbf{u}^{space} \in T\mathcal{M}(\boldsymbol{\theta})$ is admissible.

It is important to note that the spacing control lives in an $N - 2$ dimensional space, provided no two unicycles are parallel, because two degrees of freedom are needed for velocity matching. Therefore, no spacing control exists for $N < 3$. ■

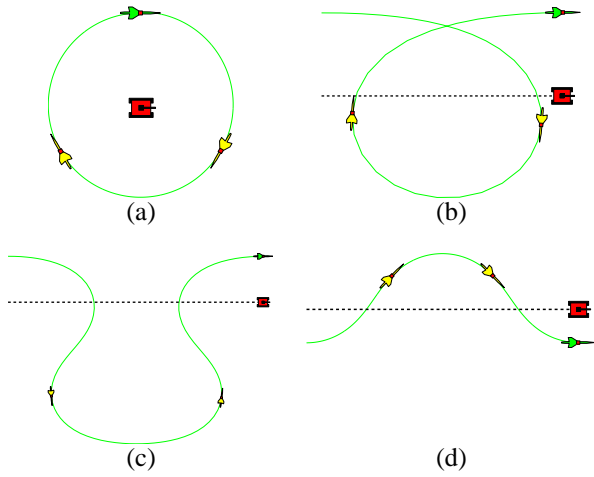


Fig. 2. Some example splay state like trajectories are shown for three vehicles with unit speed unicycle kinematics. The velocity of the tank was used as a reference and its speed was (a) $s_t = 0$, (b) $s_t = 0.3$, (c) $s_t = 0.4$, (d) $s_t = 0.8$. All three vehicles are using matching and spacing control, where the spacing component is given in (39). Notice in particular that $D_k(t, T)$ is nonincreasing, the vehicles are spaced evenly along the curve, and the centroid velocity matches the reference velocity. A control gain of $K = 0.1$ was used to generate these plots.

While it is true that any spacing control in $\ker\{J\}$ will not disturb centroid velocity matching, most of these controls will not keep individuals near the collective centroid. Only spacing controls in $\ker\{J\}$ that also satisfy $D_k(t, T) \leq 0 \forall k$ guarantees that the individuals will not stray.

The approach taken here is to look for the analogue of the *splay state*, as defined in [5]. The splay state corresponds to all vehicles tracing the same circular orbit around the collective centroid with constant and equal phase shifts between individuals. Equal phase shifts on a circular orbit necessarily implies that the centroid is stationary. Therefore, the splay state can only meet velocity matching and centroid spacing requirements for $s_{ref} = 0$.

To extend the idea of the splay state to a moving centroid, define $\Psi(t)$ to be a T -periodic continuous function with the intent of using spacing control as $u_k^{space}(t) = \Psi(t + (k-1)T/N)$. In other words, all individuals have a similar spacing control, the difference being phase shifts of equal and constant amounts (as in the splay state). Clearly, interest lies in finding $\Psi(t)$ curves that result in non-increasing centroid distances (28) and further ensure that velocity matching is not disrupted. From Theorem 3, this later requirement can be expressed as

$$[\Psi(t), \Psi(t + T/N), \dots, \Psi(t + (N-1)T/N)]^T \in \ker\{J\}. \quad (37)$$

Some examples of three vehicles in this extended notion of a splay state are shown in Fig. 2.

A necessary condition for a spacing control to satisfy this constraint is that the control must be *equivariant* with respect to ring permutations, a relabeling of vehicles that preserves neighbors. For example, $\sigma : 1234 \rightarrow 2341$ is a ring permutation of four vehicles. The need for equivariance comes from the assumption of constant and equal phase

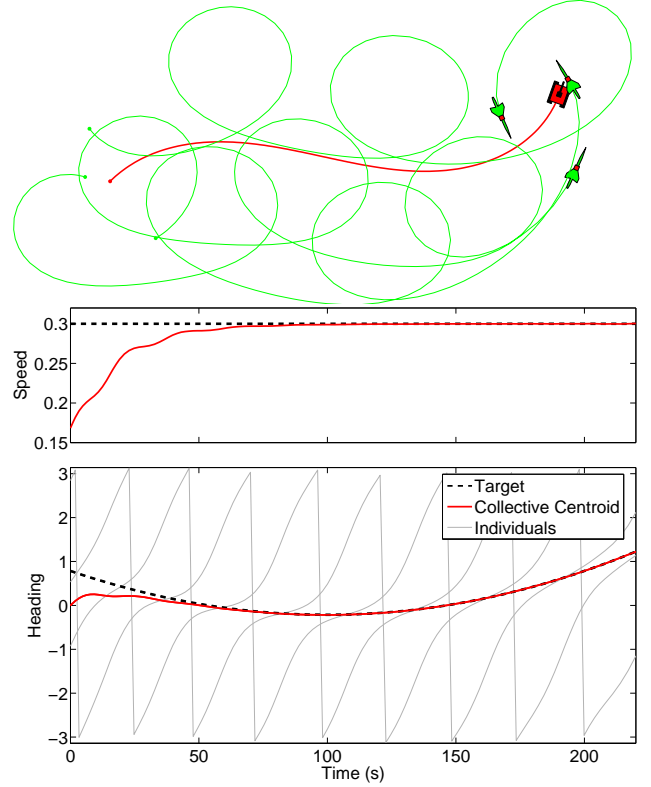


Fig. 3. Three vehicles tracking a reference velocity with unicycle dynamics. Spacing control steers the individual vehicles to stay near the collective centroid (top), while the velocity of the collective centroid matches the reference velocity (bottom). A control gain of $K = 0.1$ was used here.

shifts. For the system $\dot{\theta} = F(\theta)$, equivariance implies

$$\sigma \circ F(\theta) = F(\sigma \circ \theta), \quad (38)$$

where σ is a ring permutation. Finding spacing controls that satisfy both periodic and collective constraints for arbitrary N is quite challenging. However, for $N = 3$, an admissible spacing control law can be constructed as

$$\mathbf{u}^{space} = \begin{bmatrix} \sin(\theta_3 - \theta_2) \\ \sin(\theta_1 - \theta_3) \\ \sin(\theta_2 - \theta_1) \end{bmatrix}. \quad (39)$$

This spacing control is in the kernel of J , is equivariant, and simulation results, like the ones shown in Figs. 2 and 3, indicate that $D_k(t, T) = 0 \forall k$.

An interesting transition results from this spacing control at $s_t = (N-2)/N$. Above this critical value, the vehicles can never have a heading that is opposite from the reference velocity. Even if the other $N-1$ vehicles are aligned with the reference, one vehicle pointing the opposite direction brings the maximum speed of the collective centroid down to $(N-2)/N$. If the reference speed is less than this critical value, the vehicles can rotate all the way around in a period. This transition is demonstrated in Fig. 4.

V. OUTER LOOP CONTROL

The purpose of the outer loop controller on each vehicle is to automatically generate the reference velocity in such a way

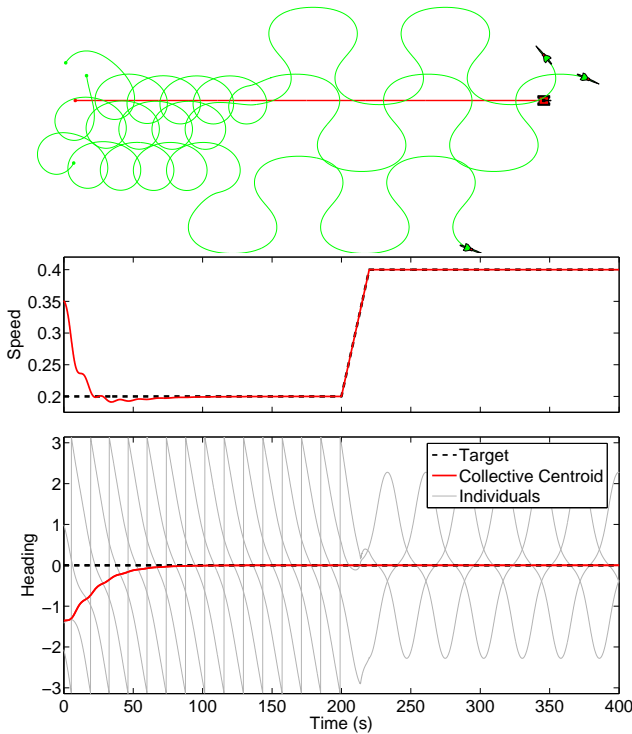


Fig. 4. Automatic transition from circling to waving as the reference speed increases from 0.2 to 0.4. A control gain of $K = 0.1$ was used here.

as to regulate spatial errors between the collective centroid and the target vehicle and to compensate for inaccuracies in the feedforward control. Conceptually, the reference velocity should point in the direction of the target when the spatial error is large, and should match the target velocity if there is no spatial error.

The transition between spatial error reduction and velocity matching can be achieved smoothly by a weighting function, $\omega(\rho)$. Here, ρ is the distance between the collective centroid and the target vehicle. The commanded centroid velocity, $\dot{r}_{ref}(t)$, should be carefully distinguished from the velocity of the target vehicle, $\dot{r}_t(t)$. The weighting function determines the commanded centroid velocity as

$$\dot{r}_{ref}(t) = (1 - \omega(\rho))\dot{r}_t(t) + \omega(\rho)(r_t - \bar{r}(\mathbf{r}))/\rho, \quad (40)$$

with the requirement $\lim_{\rho \rightarrow 0} \omega(\rho)/\rho = 0$.

Theorem 4: Outer Loop Control. Any smooth weighting function, $\omega(\rho)$, such that $0 < \omega(\rho) < 1$ and such that $\lim_{\rho \rightarrow 0} \omega(\rho)/\rho$ is zero will drive the collective centroid to the target assuming the velocity of the collective centroid matches the reference velocity (40).

Proof: Consider the Lyapunov candidate based on the position-only subsystem,

$$V(\mathbf{r}, t) = \frac{1}{2} \|r_t(t) - \bar{r}(\mathbf{r})\|^2. \quad (41)$$

The time derivative of V along the trajectories of the system is

$$\dot{V}(\mathbf{r}, t) = \langle r_t(t) - \bar{r}(\mathbf{r}), \dot{r}_t(t) - \dot{\bar{r}}(\boldsymbol{\theta}) \rangle. \quad (42)$$

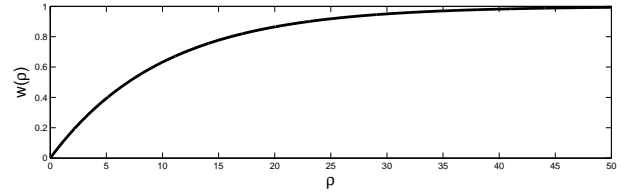


Fig. 5. The weighting function, $\omega(\rho) = 1 - e^{-\alpha\rho}$, is shown with $\alpha = 0.1$.

Using $\dot{\bar{r}}(\boldsymbol{\theta}) = \dot{r}_{ref}(t)$ and (40), gives

$$\begin{aligned} \dot{V}(\mathbf{r}, t) &= \langle r_t(t) - \bar{r}(\mathbf{r}), \dot{r}_t(t) - (1 - \omega(\rho))\dot{r}_t(t) - \\ &\quad \omega(\rho)(r_t(t) - \bar{r}(\mathbf{r}))/\rho \rangle \\ &= \omega(\rho) \langle r_t(t) - \bar{r}(\mathbf{r}), \dot{r}_t(t) - (r_t(t) - \bar{r}(\mathbf{r}))/\rho \rangle. \end{aligned} \quad (43)$$

Requiring $0 < \omega(\rho) < 1$ and $|\dot{r}_t| < 1$ ensures that the time derivative of the Lyapunov function is negative, even in the worse case of $\dot{r}_t(t) = |\dot{r}_t|(r_t(t) - \bar{r}(\mathbf{r}))/\rho$. ■

As an example, consider

$$\omega(\rho) = 1 - e^{-\alpha\rho}. \quad (44)$$

This weighting function is shown in Fig. 5. The outer loop reference command can be computed as

$$\dot{r}_{ref}(t) = e^{-\alpha\rho}\dot{r}_t(t) + (1 - e^{-\alpha\rho})(r_t(t) - \bar{r}(\mathbf{r}))/\rho. \quad (45)$$

A feedforward control term for each individual can be computed similarly to (27), but based on the outer loop reference command. The results of Theorems 2 and 3 suggest that the collective centroid velocity will converge to the reference velocity. In turn, the reference velocity will drive the collective centroid to the target, as in Theorem 4. See Fig. 6 for simulation results.

VI. CONCLUSIONS AND FUTURE WORK

The work in this paper has addressed the general class of problems in which a group of constant speed nonholonomic vehicles is required to track a reference vehicle, real

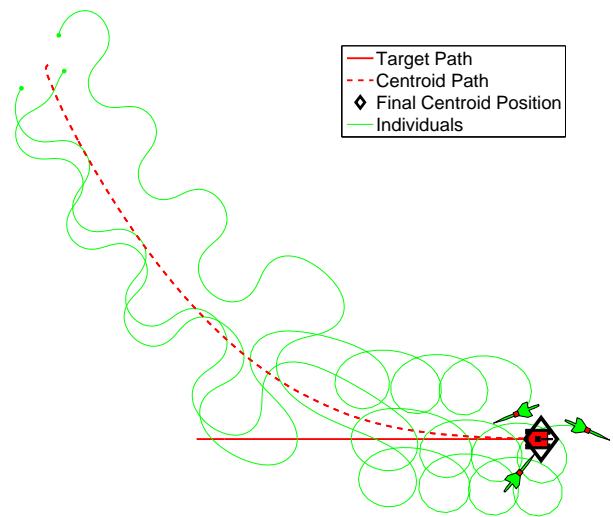


Fig. 6. Outer loop simulation with $K = 0.1$ and $N = 3$. The outer loop drives the collective centroid to the target vehicle. Spacing control is enabled so the individual vehicles do not stray far from the collective centroid.

or virtual. The approach taken was built upon new and existing theories of oscillator models. First, an alignment steering control was derived such that the velocity of the collective centroid would approach a reference velocity. Then a framework for spacing control was outlined and the specific example of $N = 3$ was demonstrated. Finally, an outer loop controller was added to achieve tracking of a target vehicle.

These results are a significant advancement of previous work on trajectory planning for fixed wing UAVs. Prior work was ad-hoc and only considered a single UAV. In contrast, the work in this paper has mathematical underpinnings and is applicable to any number of constant speed vehicles.

The results here indicate that trajectory tracking can be achieved with a group of N unit speed unicycles. However, many questions remain unanswered. For example, the spacing control for more than three vehicles remains an open problem. Another area to consider in future work is the network structure of the problem. This paper considered all-to-all communication without delay or noise. However, communication range, delay, and noise are all present in real systems. Time delay [16], [17], [18] and network topology [10] have been studied in the phase coupled oscillator literature, however the results need to be extended to include a reference vehicle. Immediate future work will deal with these issues.

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