

On Controlled Sinusoidal Phase Coupling

Daniel J. Klein, Emmett Lalish, and Kristi A. Morgansen

Abstract—The work in this paper considers a heterogeneous coordinated phase control problem in which most agents follow a classic sinusoidal coupling protocol, but a select few agents act as leaders. These leaders have the ability to report some value other than their current phase to their neighbors. This heterogeneous model is motivated by systems in biology and in engineering. In biological contexts, observations have been made that groups of trained animals can bias the behavior of a much larger groups of untrained animals. Translating these results to engineered contexts is of interest to, for example, reduce the number of human operators necessary to control a fleet of many autonomous agents. The contributions of this paper include a general reachability result, a proof that a chain of four or more agents is uncontrollable by a single leader, and a nonlinear controllability analysis of some example problems. An interesting result is that symmetry about the leader node is not sufficient to guarantee uncontrollability of the follower nodes, as it is in the related controlled linear agreement problem.

I. INTRODUCTION

The design and implementation of coordinated controllers for distributed autonomous multi-agent systems is a fundamental challenge that has received much interest over the past several years. Principle motivations for this interest are that multi-agent teams offer the possibility of increased performance through parallelization, better chances of success through redundancy, and the ability to perform coordinated tasks that were otherwise impossible to achieve with a single agent. These benefits, however, can be difficult to harvest due to the increased complexity of multi-agent systems. The thesis of this paper is that a select few (leader) agents can, under some topologies, control all other (follower) agents, effectively reducing complexity through heterogeneity.

To gain insight into how to design distributed controllers for coordinated multi-agent systems, control theorists have turned to well-studied models of collective behavior. One such model created by Kuramoto [1], describes synchronization in large groups of oscillators that are coupled sinusoidally in phase. Indeed, much attention has been given to understanding the Kuramoto model (see [2] for a good review). Most recently, researchers have applied insights gained from studies of the Kuramoto model to distributed control of autonomous agents. Early work with steering control inputs for constant speed unicycle vehicles [3] has been linked to the Kuramoto model [4]. Recent work has built extensively upon this foundation [5], [6], [7], although these studies have only considered homogeneous agent dynamics.

This work was supported in part by NSF grant CMS-0238461. The authors are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98195-2400. {djklein,emmett1,morgansen}@aa.washington.edu.

A promising direction for distributed control of autonomous multi-agent systems considers a situation in which one or more of the agents do not obey the usual control law. Unlike classical leader/follower control design, the followers do not know which agents are leaders and thus treat all neighbors equivalently. The leader agents have the potential to report some value other than their current phase to their neighbors, but each neighbor receives the same value. The prototypical example is the controlled linear agreement problem [8], [9], [10], [11], in which follower nodes apply a linear consensus protocol. A key result is that the ability of a single leader node to control the other nodes is dependent on the topology of the communication network. In particular, if the topology is symmetric about the leader nodes, the states of the other nodes are not controllable in the typical linear systems sense.

A parallel research theme can be found in biology, where researchers are working to understand how heterogeneity and topology influence the behavior of large aggregations. Ongoing research with heterogeneous aggregations of Giant Danio (*Devario aequipinnatus*) suggests that as few as three trained (i.e. leader) fish are required to make twelve untrained (i.e. follower) fish behave as if they were trained [12].

In this paper, the theme of heterogeneity through leadership is extended from linear consensus to situations in which the follower agents obey a nonlinear (sinusoidal) phase coupling protocol. The main contributions are as follows. The dynamics of the controlled sinusoidal coupling problem are first written for an arbitrary interconnection topology and leadership node assignment. A basic result showing that the aligned set is reachable from initial conditions in a common semicircle is established in the first theorem. Then, the dynamics are rewritten in several ways to permit a complete but informal analysis of three specific example problems on three nodes. These simple examples are intended to highlight the main differences between controlled linear and controlled sinusoidal protocols. The three-node chain example generalizes to N nodes to reveal that a chain of four or more agents is never controllable by a single leader. An interesting conclusion is that symmetry about the leader node does not imply uncontrollability of the follower nodes.

The material presented here is organized as follows. In the next section, the problem is formulated and mathematical preliminaries are introduced. In Section III, a basic reachability analysis is presented for the controlled sinusoidal coupling problem. Analysis and simulation of specific example topologies are presented in Section IV. Also included at this point is an analysis of a N length chain. Conclusions are presented in Section V.

II. PROBLEM FORMULATION

The notation, conventions and assumptions used throughout this work are described in this section. Also, leader and follower subgraphs are defined and the definitions reachability and controllability are reviewed.

A. Leader and Follower Subgraphs

A graph $G = (V, E)$ is a set of nodes, V , and edges, $E \subseteq V \times V$, in which each edge connects one node to one other node and no two nodes are joined by more than one edge. The cardinality of the node and edge sets are denoted $|V|$ and $|E|$, respectively. An undirected graph is said to be connected if a path exists from every node to every other node, whereas a directed graph is connected if a directed path exists from one node to every other node. Throughout this work, all graphs are assumed to be undirected and connected, unless otherwise noted.

Associating an arbitrary direction with each edge, the directed incidence matrix, B , of a graph G is a $|V| \times |E|$ matrix defined as

$$B(i, j) = \begin{cases} 1 & \text{if edge } j \text{ leaves node } i \\ -1 & \text{if edge } j \text{ enters node } i \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The incidence matrix has rank $|V| - 1$ whenever the graph is connected [13]. The neighbors of node i , \mathcal{N}_i , are the set of nodes that are adjacent to v_i in G . In a complete graph, every node is adjacent to every other node.

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a graph with nodes $V' \subseteq V$ and edges $E' \subseteq E$. An induced subgraph $G'(V')$ of graph G is formed by keeping only edges of G that connect nodes in V' to other nodes in V' .

An interacting group of N agents can be modeled as a graph $G = (V, E)$ in which nodes with associated dynamics represent agents, and edges represent inter-agent communication. For the work in this paper, we consider two types of nodes: leader nodes and follower nodes. Thus, the node set can be partitioned into leader and follower node sets, V_L and V_F , respectively. Decomposing G into corresponding subgraphs will prove useful later in this work.

Definition 2.1 (Follower Subgraph): Define the follower subgraph G_F to be the subgraph induced by the follower nodes, V_F .

Definition 2.2 (Leader Subgraph j): Let $v_j \in V_L$ be a leader. The subgraph corresponding to this leader node is defined as

$$G_j = (V, E_j) \quad (2)$$

$$E_j = \{(v_i, v_j) \in E | v_i \in V_F\}. \quad (3)$$

In other words, G_j consists of all nodes, but only retains edges from E that connect leader node j to its neighboring nodes that are followers.

Denote by B_F and B_j any directed incidence matrix associated with the follower and j^{th} leader subgraphs. Note that edges connecting one leader to another do not appear in either leader or follower subgraphs.

B. Reachability and Controllability

Some basic definitions and tools from the theory of nonlinear control will prove useful in the later parts of this paper. In particular, the dynamics considered here can be written in standard control-affine form,

$$\dot{x} = f_0(x) + \sum_{i=1}^m f_i(x)u_i. \quad (4)$$

The first vector field, f_0 , is called the drift of the system. The other m vector fields are control vector fields and u_i is the i^{th} control input. Unlike the drift vector field, the control vector fields can be reversed or nulled entirely through the choice of inputs.

The set of all states attainable in exactly time T from a point x_0 by any control input is the T -reachable set from that point, denoted $\mathcal{R}(x_0, T)$. The set of all states that can be eventually reached from x_0 is called the reachable set, $\mathcal{R}(x_0)$. A system is said to be controllable from x_0 if every other point in the domain is reachable from x_0 (i.e. $\mathcal{R}(x_0) = \mathcal{D}$). Finally, a system is small time locally controllable at a point x_0 if there exists a $T > 0$ such that $x_0 \in \mathcal{R}(x_0, t)$ for each $t \in (0, T]$ [14].

C. The Sinusoidally-Coupled Oscillator Model

Consensus seeking controllers often have the form

$$\dot{x}_i = \sum_{j=\mathcal{N}_i} f(x_j - x_i). \quad (5)$$

Here, x_i is the state of the i^{th} agent and f is a coupling function. The basic intuition is that for $y^T f(y) > 0$, $y \in \mathbb{R} \setminus \{0\}$, each agent moves towards the average of its neighbors' states. Connectivity information from the graph incidence matrix can be used to rewrite (5) as

$$\dot{x} = -Bf(B^T x). \quad (6)$$

When f is a constant, the protocol is linear. Indeed, much is known about linear consensus protocols, for which the system dynamics can be written compactly as

$$\dot{x} = -Lx, \quad (7)$$

where $L = BB^T$ is the graph Laplacian. When G is connected and $f > 0$, the span of the vector of ones, called the agreement subspace, is globally attracting.

The use of linear consensus protocols is most appropriate on linear spaces, like \mathbb{R}^N . For the work in this paper, the state of the N -agent system is a point $\theta \in \mathbb{T}^N$, for instance representing the headings of the agents. A well-studied coupling function that is suited to this space is $f(\cdot) = \sin(\cdot)$, which yields

$$\dot{\theta} = -B \sin(B^T \theta). \quad (8)$$

Intuitively, each agent steers to become more aligned (locally) with its neighbors.

Much that is known about sinusoidal coupling comes from work in physics, chemistry, and biology in which the Kuramoto model,

$$\dot{\psi}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\psi_j - \psi_i). \quad (9)$$

has been studied for years. Here, $\psi \in \mathbb{T}^N$ is the state and $\omega_i \in \mathbb{R}$ is the natural frequency of the i^{th} “oscillator”. Frequency synchronization occurs when the coupling strength, K , is sufficiently strong.

For engineering purposes, it makes sense to choose homogeneous natural frequencies, in which the Kuramoto model (9) reduces to all-to-all sinusoidal phase coupling through the state transformation $\theta_i = \psi_i - \omega_i t$. With all-to-all communication, the aligned set,

$$\mathcal{A} = \left\{ \theta \in \mathbb{T}^N \mid \left\| \frac{1}{N} \sum_{i=1}^N \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} \right\| = 1 \right\}, \quad (10)$$

is almost globally attracting for $K > 0$. The balanced (i.e. anti-aligned) set can be defined similarly,

$$\mathcal{B} = \left\{ \theta \in \mathbb{T}^N \mid \left\| \frac{1}{N} \sum_{i=1}^N \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} \right\| = 0 \right\}. \quad (11)$$

While these stability results require all-to-all connectivity, recent work has explored more general topologies [15], [16]. An important result is that the aligned set is attracting when all phases are initially in a common semicircle, however the aligned set is known not to be globally attracting [17].

III. DYNAMICS AND ALIGNED SET REACHABILITY

In controlled phase coupling, a subset of the nodes (called leader nodes) have the ability to transmit a control signal to their neighbors. The follower (i.e. non-leader) nodes do not know of the leaders existence, and thus process received information according to the originally prescribed model (8). All neighbors of a particular leader receive the same value from that leader. We begin by writing the system dynamics.

Let $\phi \in \mathbb{T}^{|V_F|}$ be the phases of the follower agents (the phase of the leader nodes is not used, and therefore excluded). As in previous work with controlled linear consensus, the leader nodes have access to all follower phases. The phase rate of the i^{th} follower agent can then be written as

$$\dot{\phi}_i = \sum_{j \in \mathcal{N}_i} \begin{cases} \sin(u_j - \phi_i), & \text{if } j \text{ is a leader} \\ \sin(\phi_j - \phi_i), & \text{otherwise.} \end{cases} \quad (12)$$

Here, u_j is the control signal sent out by leader node j . Using the subgraphs defined above, the follower node dynamics (8) can alternatively be written as

$$\dot{\phi} = -B_F \sin(B_F^T \phi) - \sum_{i=1}^{|V_L|} P^T B_i \sin(B_i^T P(\phi - u_i \mathbf{1})). \quad (13)$$

The matrix P is formed by selecting columns corresponding to follower node indices from an $N \times N$ identity matrix.

The following lemma is the first contribution of this work, and is required for the subsequent theorem.

Lemma 3.1: Consider a connected graph $G = (V, E)$ on N nodes and select any one node as a leader. The (entire) aligned set is *reachable* from any point in the aligned set.

Proof: Assume without loss of generality that the first node is selected as the leader and that the followers have initial state $\phi(0) = \alpha \mathbf{1} \in \mathcal{A}$, $\alpha \in \mathbb{T}$. To show that the aligned set is reachable, select any $\beta \in \mathbb{T}$ as a goal point. If there exists a leader controller taking the follower state from $\phi(0) = \alpha \mathbf{1}$ to $\lim_{t \rightarrow \infty} \phi(t) = \beta \mathbf{1}$, the aligned set is reachable.

Consider a constant leader controller, $u(t) = \beta$, $\forall t \geq 0$. This choice permits the overall heterogeneous system dynamics, including both leader and follower nodes, to be viewed as a certain homogeneous system. In particular, the equivalent homogeneous system consists of N nodes, each of which applies sinusoidal coupling to all incident edges as in (8). Edges in the follower subgraph are undirected, as usual, but edges in the leader subgraph are *directed*, going from the leader to neighbors. The state of the leader node in the equivalent homogeneous system never changes because it has no incident edges whereas the follower nodes will behave as they would in the original heterogeneous system.

Then, a recent result by Moreau [18] can be leveraged to conclude that the state of the equivalent homogeneous system will asymptotically approach $\beta \mathbf{1}$, and hence the state of follower nodes in the original heterogeneous system must also approach $\beta \mathbf{1}$. The main idea of Moreau’s proof is that the convex hull of the state decreases to a singleton, under some connectivity assumptions. The equivalent homogeneous system with directed topology meets these connectivity assumptions because a directed path exists from the leader node to each follower node, by construction.

Moreau’s proof is designed for Euclidean spaces, but Example 2 of [18] shows how the result can be applied systems with state in \mathbb{T}^N provided all phases are within a common semicircle. Here, the followers all start at α , so the result can be used directly provided $\beta \neq \alpha + \pi$. To show that $\beta = \alpha + \pi$ is also reachable, the leader can temporarily report $u = \alpha + \pi/2$ and later change to $u = \beta$ once all agents have left α . ■

The following theorem is a main contribution of this work.

Theorem 3.2: Consider a connected graph G on N nodes, and select any one node as a leader. Then, for initial follower phases in a common semicircle, (i.e. $\phi - \phi_0 \in (-\pi/2, \pi/2)^{|V_F|}$ for some $\phi_0 \in \mathbb{T}$), the aligned set is always reachable.

Proof: Proof is by construction of a controller. Without loss of generality, assume the initial state of the followers is $\phi \in (-\pi/2, \pi/2)^{|V_F|}$. In leaderless sinusoidal phase coupling, the aligned set is globally asymptotically stable over any compact subset of $(-\pi/2, \pi/2)^{|V_F|}$ for arbitrary connected graphs [15]. Thus, let the leader obey the usual

phase coupled oscillator model with internal state ξ ,

$$\dot{\xi} = \sum_{j \in \mathcal{N}_{\text{leader}}} \sin(\phi_j - \xi) \quad (14)$$

and initial phase $\xi \in (-\pi/2, \pi/2)$. The phase reported by the leader is its internal state, $u = \xi$. Then, a point in the aligned set will be approached [15]. Finally, the result of Lemma 3.1 can be used to show reachability of the aligned set. ■

Corollary 3.3 (Additional Leaders): Adding more leaders does not decrease the size of the reachable set, because the extra leaders can always implement follower-like behavior.

IV. EXAMPLES

Some specific examples of leader-controlled phase coupling and a general result for chain graphics are presented in this section to gain insight into this challenging problem and to highlight the main differences between controlled sinusoidal and controlled linear protocols. The first three examples are for three agents, whereas the final example consider a general chain of length N , see Fig 1.

A. Three Node Star Graph with Leader at Center

The first topology studied here is the star graph, with a single leader at the center node (see Fig. 1(a)). This topology is of interest because it is completely symmetric about the leader node. This symmetry means that controlled linear consensus is uncontrollable because the two followers cannot be controlled independently. Interestingly, this uncontrollability result does not hold with the sinusoidal coupling studied here.

With this topology, the dynamics of each follower (12) reduce to

$$\begin{aligned} \dot{\phi}_i &= \sin(u - \phi_i), \quad i = 1, 2 \\ &= \sin u \cos \phi_i - \cos u \sin \phi_i, \end{aligned} \quad (15)$$

where $u \in \mathbb{T}$ is the phase reported by the leader. Then, the system dynamics can be rewritten as

$$\dot{\phi} = A(\phi)q(u), \quad (16)$$

with

$$A(\phi) = \begin{bmatrix} -\sin \phi_1 & \cos \phi_1 \\ -\sin \phi_2 & \cos \phi_2 \end{bmatrix} \quad \text{and} \quad q(u) = \begin{bmatrix} \cos u \\ \sin u \end{bmatrix}. \quad (17)$$

The leader can choose u to make the unit vector $q(u)$ point instantaneously in any direction. Thus, provided $A(\phi)$ is full rank, the leader can drive the state in any direction in the state space (\mathbb{T}^2). For this system, $A(\phi)$ loses rank only when the state is aligned or is balanced ($\phi \in \mathcal{A} \cup \mathcal{B}$).

When the state is aligned, the range of $A(\phi)$ is spanned by $[1, 1]^T$, which is also a basis vector for the aligned set. Thus, no control signal from the leader can eject the state from the aligned set. On the other hand, when the state is balanced, the range of $A(\phi)$ is spanned by $[1, -1]^T$. Control can drive the state out of the balanced set, but no motion is possible directly along the balanced set.

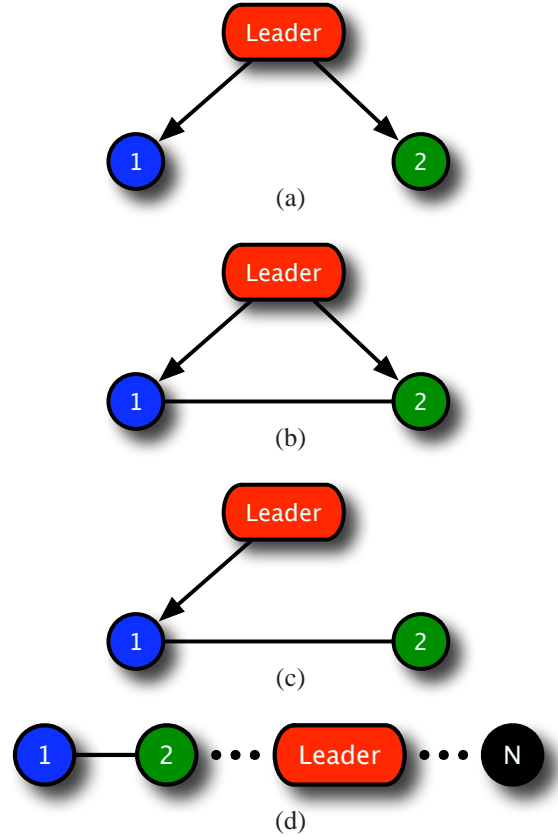


Fig. 1. The example topologies considered in this section are (a) a $N = 3$ star with the leader in the middle, (b) a $N = 3$ complete graph, (c) a $N = 3$ chain with the leader at one end, and (d) a chain of length $N > 3$ with the leader at any one node. Edges from the leader are directed indicating the follower is coupled sinusoidally to the leader, but not vice versa.

Put together, these results imply that the system is controllable from $\mathbb{T}^2 \setminus \mathcal{A}$. Because of the structure of \mathbb{T}^2 , the aligned set \mathcal{A} does not form a barrier as it would in \mathbb{R}^2 . Instead, to get from one state to another, a controller can always choose a path that does not pass through the aligned set.

Knowing that this system is controllable outside the aligned set, a simple controller can be constructed to drive the state from an initial position, $\phi(0) \in \mathbb{T}^2 \setminus \mathcal{A}$ to a goal state, $\phi^* \in \mathbb{T}^2$. The basic idea is to choose the control direction that yields the quickest reduction in distance between the current state and the goal,

$$u = \underset{\tilde{u}}{\operatorname{argmax}} ((\phi^* - \phi)^T A(\phi)q(\tilde{u})). \quad (18)$$

This optimization problem can be solved in closed form by examining the first-order necessary conditions,

$$\begin{aligned} 0 &= \frac{\partial}{\partial u} \left((\phi^* - \phi)^T A(\phi) \begin{bmatrix} \cos u \\ \sin u \end{bmatrix} \right) \\ &= (\phi^* - \phi)^T A(\phi) \begin{bmatrix} -\sin u \\ \cos u \end{bmatrix}, \end{aligned} \quad (19)$$

which are satisfied when

$$q(u) = \pm A(\phi)^T (\phi^* - \phi). \quad (20)$$

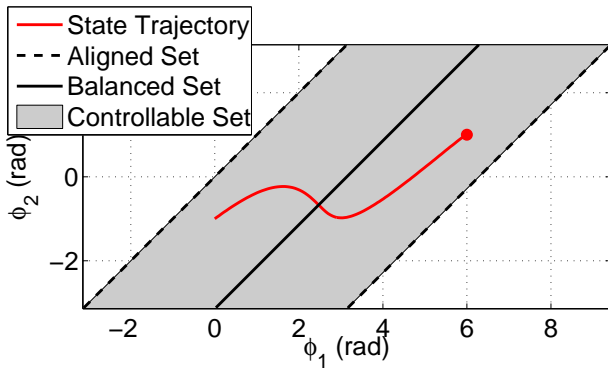


Fig. 2. The controlled state trajectory is shown for a simulation of the three node star graph with the leader at the center. The initial state is $[0, -1]$ and the goal state, shown as a red dot, is $[6, 1]$. The state space, \mathbb{T}^2 , is represented as the area between the two aligned sets (which are actually the same, as per the definition of a torus).

The positive sign is chosen so that the control takes the state in the correct direction:

$$\begin{aligned} (\phi^* - \phi)^T \dot{\phi} &= (\phi^* - \phi)^T A(\phi)q(u) \\ &= q(u)^T q(u) \\ &\geq 0. \end{aligned} \quad (21)$$

Then, the leader control can be calculated as

$$u = \text{atan}(q(u)), \quad (22)$$

where atan is the four quadrant arctangent function.

Because $A(\phi)$ is full rank outside $\mathcal{A} \cup \mathcal{B}$, the distance between the current and goal states decreases along controlled trajectories on this subset. The state does not start aligned and the controller will not drive the state into the aligned set, by construction, so the only possible trouble spot is the balanced set. Indeed, this particular controller is imperfect in that it is unable to drive the state to a goal in the balanced set. Because the system is controllable, a different controller should be used when the goal state is balanced.

To demonstrate the controller in simulation, the initial and final states were chosen at $\phi(0) = [0, -1]$ and $\phi^* = [6, 1]$. Results are shown in Figs. 2 and 3. The goal state is reached in about 7sec. Note that outside the aligned and balanced sets, no leader control input can zero the state derivative. Instead, once the goal point is reached, the controller naturally oscillates back and forth to keep the state arbitrarily close to the goal.

B. Three Node All-to-All with One Leader

The second example considered here is a complete graph on three nodes with a single leader (see Fig. 1(b)). As with the star topology, this structure is symmetric, but has a non-empty follower subgraph that creates a non-zero drift vector field. Following the analysis technique of the previous example, the dynamics of the followers can be written as

$$\frac{d}{dt} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin(\phi_2 - \phi_1) + \sin(u - \phi_1) \\ \sin(\phi_1 - \phi_2) + \sin(u - \phi_2) \end{bmatrix}. \quad (23)$$

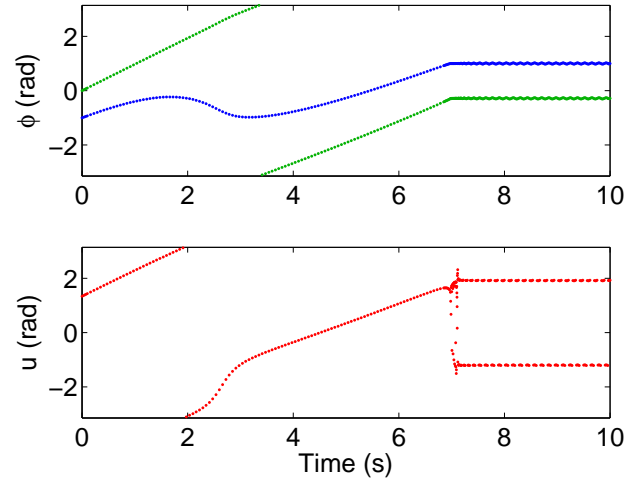


Fig. 3. Follower states ϕ (top) and leader control u (bottom) versus time. The initial state is $[0, -1]$ and the final (goal) state is $\phi = [6, 1]$. Notice how the control oscillates to keep the state near the goal.

Equivalently,

$$\dot{\phi} = A(\phi)q(u) + D(\phi), \quad (24)$$

with

$$A(\phi) = \begin{bmatrix} -\sin \phi_1 & \cos \phi_1 \\ -\sin \phi_2 & \cos \phi_2 \end{bmatrix} \quad (25)$$

$$D(\phi) = \begin{bmatrix} \sin(\phi_2 - \phi_1) \\ -\sin(\phi_2 - \phi_1) \end{bmatrix}. \quad (26)$$

Just as with the star graph, $A(\phi)$ is full rank everywhere except on the aligned and balanced sets. Thus, outside $\mathcal{A} \cup \mathcal{B}$, the leader control u can push the state in any direction in the state space. However, the magnitude of this control is limited by the fact that $q(u)$ is unit norm. The drift D pushes the system toward alignment (as expected, because all-to-all coupling is almost globally stable to alignment). Thus, the system is only controllable when the magnitude of the control is large enough to overcome the drift. Once the state is sufficiently close to the aligned set, alignment cannot be prevented by any control.

To explicitly determine the subset of the domain on which this system is controllable, the control design technique from the star graph example can be employed. Instead of calculating which input vector $q(u)$ yields the greatest velocity towards the goal (20), we determine which input vector results in the greatest velocity against the drift,

$$\begin{aligned} q(u) &= A(\phi)^T D(\phi) \\ &= \sin(\phi_2 - \phi_1) \begin{bmatrix} \sin \phi_2 - \sin \phi_1 \\ \cos \phi_1 - \cos \phi_2 \end{bmatrix}, \end{aligned} \quad (27)$$

where u is still calculated from (22). Physically, this choice of input corresponds to the leader reporting that it is located across the phasor circle from the average of the followers' phasors. With this choice of input, the drift will overwhelm the control when $\cos(\phi_2 - \phi_1) < -1/2$. In other words, the leader is only effective at overcoming the drift when the

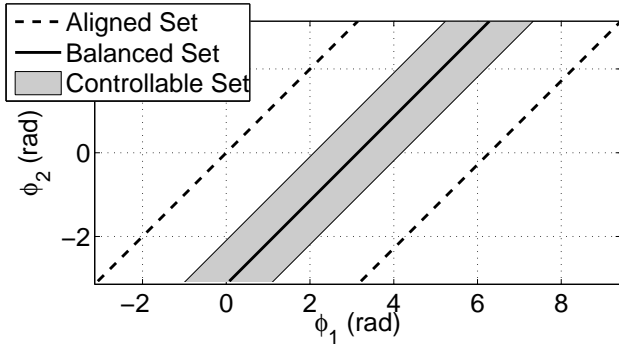


Fig. 4. The controllable set on \mathbb{T}^2 is shown for the complete graph. When the angle between the followers is less than 120° , alignment cannot be prevented.

followers are separated by at least 120° . Thus, this system is only controllable inside a band around the balanced set, as shown in Fig. 4.

C. Three Node Chain with Leader at End

The next topology considered here is a three-node chain with the leader at one end (see Fig. 1(c)). The main difference between this example and the previous ones is the lack of symmetry about the leader. Again, the edge between the followers results in a non-zero drift vector field. The dynamics in this case are

$$\frac{d}{dt} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin(\phi_2 - \phi_1) + \sin(u - \phi_1) \\ \sin(\phi_1 - \phi_2) \end{bmatrix}. \quad (28)$$

For this topology, rewriting the dynamics in terms of $A(\phi)$ and $q(u)$ does not help because $A(\phi)$ is never full rank. Instead, consider the coordinate transformation $\bar{\phi} = \phi_1 + \phi_2$, $\tilde{\phi} = \phi_1 - \phi_2$ and $v = \sin(u - \phi_1)$, for which

$$\frac{d}{dt} \begin{bmatrix} \bar{\phi} \\ \tilde{\phi} \end{bmatrix} = \begin{bmatrix} v \\ -2\sin(\tilde{\phi}) + v \end{bmatrix}, \quad (29)$$

and $v \in [-1, 1]$. The control is found by $u = \arcsin(v) + \phi_1$, but when the sine of the angle difference is larger than one half, no u can be found to counter the drift of the system. Thus, the drift dominates the control and drags the state towards the aligned set when $30^\circ < |\tilde{\phi}| < 150^\circ$.

With this result in mind, the state space can be partitioned into three sets,

$$\mathcal{S}^C = \{\phi \in \mathbb{T}^2 \mid 150^\circ < |\phi_1 - \phi_2|\} \quad (30)$$

$$\mathcal{S}^D = \{\phi \in \mathbb{T}^2 \mid 30^\circ < |\phi_1 - \phi_2| \leq 150^\circ\} \quad (31)$$

$$\mathcal{S}^R = \{\phi \in \mathbb{T}^2 \mid |\phi_1 - \phi_2| \leq 30^\circ\}. \quad (32)$$

For points in the controllable set (\mathcal{S}^C), every other point in \mathbb{T}^2 is reachable in finite time, and thus the system is controllable from this set. Once the state enters the drift dominated set (\mathcal{S}^D), the above analysis shows that entering the reachable set (\mathcal{S}^R) is unavoidable. Every point in the reachable set is reachable from every other point in the set. However, the reachable set is positively invariant, meaning it cannot be escaped. The controllable, drift dominated, and

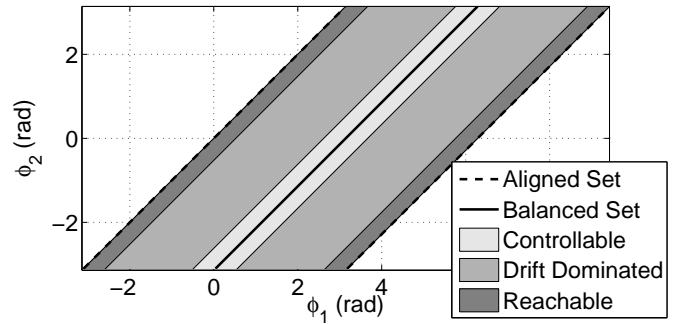


Fig. 5. The sets for the chain graph are shown. All states are reachable from the controllable set (light gray), however the positively invariant reachable set (dark gray) cannot be avoided after entering the drift dominated set (medium gray).

reachable sets are depicted in Fig. 5 as light, medium, and dark gray shaded regions, respectively.

D. $N > 3$ Node Chain with One Leader

While regions of reachability and controllability have been demonstrated for the three node chain, the fact remains that the follower agent phases are uncontrollable on the 2-torus (i.e. not controllable everywhere). This result can be extended to a chain of length $N > 3$ with a single leader, see Fig. 1(d).

Corollary 4.1 (N-Agent Chains): Consider a group of $N > 3$ agents connected in a chain and choose any one node as the leader. Then, the leader is unable to control the state of the $N - 1$ follower nodes on the domain.

Proof: Every chain of length $N > 3$ necessarily terminates at one end with two adjacent follower nodes, independent of the location of the leader in the chain. Even if the leader were able to directly control the node adjacent to these two nodes, the result of the three-node chain example from Section IV-C shows that the phases of the two end nodes is uncontrollable. Thus, the phase of the $N - 1$ follower nodes is also uncontrollable. ■

This result is contrary to the controlled linear consensus problem, where the N -agent chain is controllable when the topology is not symmetric (e.g. the leader is at one end). Note that as in the three-node example, there may exist controllable regions and the aligned set is always reachable from a semicircle, however the overall uncontrollability of the chain is universal and any controllable region is likely reduced in volume as nodes are added to the chain.

V. CONCLUSION

The work in this paper has extended previous work on linear controlled agreement to sinusoidal coupling on the N -torus. While much remains to be learned about this fascinating system, results presented here show that, for any connected topology with one or more leaders, the aligned set is reachable from any initial state in a hemisphere. Specific examples were then presented and analyzed to demonstrate the effect of topology and the significant differences between

linear and sinusoidal couplings. One observation is that leader symmetry does not imply uncontrollability, as it did in the linear case.

Future work will build upon the foundation presented here to answer more general questions of reachability and controllability. For instance, if a single leader can be placed anywhere in a large network, where should it be put in order to be most effective? Also, how few leader nodes are required to make the heterogeneous system controllable?

To answer these questions and make additional progress on this problem, future work will connect this problem to existing theory and develop new tools as necessary. Control of nonlinear systems that contain drift is an active area of research.

Future work will also focus on closing the loop with biologists to see if controlled phase coupling is a good model of heterogeneity in natural aggregations. If so, effort will be directed towards inter-disciplinary work in this area.

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