Observability Optimization for the Nonholonomic Integrator

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Abstract—A growing number of control problems are concerned with vehicles that have insufficient sensors to directly measure all system states and require active control actuation to observe all necessary states (e.g., underwater navigation, simultaneous localization and mapping, parameter identification). In this paper, we consider first-order nonholonomic systems in canonical form as a model system for exploring the development of active control that optimizes system observability characteristics. Observability analysis of this system shows that all nonholonomic states must be measured in the output, and actuation in a minimum of two control channels is required for observability. Analytical trajectories are derived that allow almost arbitrary placement of the observability gramian eigenvalues, which are shown to be inversely related to state estimation covariances. Simulation results show that the optimal trajectories provide four to five times faster estimator convergence and three times lower state estimate covariances than suboptimal trajectories.

I. INTRODUCTION

The crux of many guidance, navigation, and control problems is the ability to estimate the system states \( x(t) \) from the time history of output \( y(t) \). Observability analysis provides engineers with a tool to determine if the system states can be identified and to what degree each state can be observed. For linear systems, the observability analysis is independent of the control input, thus if the system has poor observability characteristics, the only avenue for improvements is modifying the output equation (i.e., add more sensors) or modifying the dynamics (i.e., redesign the system). For nonlinear systems, however, the sensing and actuation may be coupled in the observability analysis, and the choice of control input can directly affect the system observability characteristics. This coupling can be exploited to actively shape the system observability.

A growing number of guidance, navigation, and control problems are concerned with under-sensed systems—that is, systems with insufficient sensors to directly measure all necessary states and systems that require actuation for observability. For example, the simultaneous localization and mapping (SLAM) problem requires active motion of the vehicle to track and observe landmarks [1], [2]. Underwater vehicle navigation using a single or two beacons also requires careful path planning to achieve observability [3]. Parameter identification problems and mobile sensor networks fall into this class of problems as well.

In this paper, we explore the coupling of sensing and actuation for the nonholonomic integrator and derive controls that optimize the system observability characteristics. This class of systems has practical importance for several reasons. First, these systems exhibit fundamental challenges due to the nonholonomic constraints which form the most basic, but interesting nonlinearity. Second, many systems of engineering interest are diffeomorphic to the nonholonomic integrator, such as the unicycle model for robotic navigation. Finally, the properties of the nonholonomic integrator give insight into systems with higher complexity.

Previous work by the authors of this paper derived analytical trajectories that optimize the observability of uniform flow fields in the context of underwater and air vehicle navigation [4], [5]. Other researchers have utilized the numerical approach of the empirical observability gramian [6] to optimize observability of spatially varying flow fields using multiple vehicles [7] and to optimize sensor placement on an airfoil for flow identification [8].

The contributions of this paper are threefold. First, observability analysis of the nonholonomic integrator is given. Second, the properties of the observability gramian eigenvalues are further explored, including their relation to estimation covariances. Finally, an analytical solution to the observability optimization problem for this class of systems is derived.

The remainder of the paper is organized as follows. A brief review of nonlinear controllability and observability, and a detailed observability analysis of the nonholonomic integrator is provided in Section II. Possible metrics for measuring observability are discussed in Section III-A. Observability-optimal controls for the nonholonomic integrator are derived in Section III-B. A sample calculation of the optimal controls for desired observability characteristics are provided in Section III-C. Finally, the power of observability optimization is demonstrated through simulation results in Section IV, and some discussion and concluding remarks are given in Section V.

II. NONHOLONOMIC INTEGRATOR

Before describing the system dynamics in detail, some basics of nonlinear controllability and observability are reviewed. For more detailed information, the reader is referred to [9]. Here we will consider driftless, nonlinear, control-affine systems of the form

\[
\dot{x} = \sum_{i=1}^{m} f_i(x)u_i, \\
y = h(x),
\]
where \(x \in \mathbb{R}^n\) are the states, \(u \in \mathbb{R}^m\) are the controls, and 
\(y \in \mathbb{R}^p\) are the outputs. Define the Lie bracket of two vector fields as

\[
[f_1, f_2](x) = \frac{\partial}{\partial x} f_2(x) f_1(x) - \frac{\partial}{\partial x} f_1(x) f_2(x). \tag{2}
\]

Lie brackets of control vector fields can be generated by area-generating or switching controls, which produce new control “directions” in which the system can be driven. The control Lie algebra, \(\mathcal{C}\), spans all the control vector fields \(f_i(x)\) and their Lie brackets \(\{f_i, f_j\}(x), \{f_i, f_j, f_k\}(x), \ldots\). By Chow’s theorem, if \(\dim(\mathcal{C}) = n\), then system (1) is said to be small time locally controllable.

The Lie derivative of the output function, \(h(x)\), with respect to a vector field, \(f_i(x)\) is defined as

\[
L_{f_i} h = \frac{\partial h}{\partial x} f_i.
\tag{3}
\]

Repeated Lie derivatives are subsequently computed by \(L_{f_i}^k h = \frac{\partial}{\partial x} (L_{f_i}^{k-1} h) f_i\), and mixed Lie derivatives are computed by \(L_{f_i} f_j h = \frac{\partial}{\partial x} (L_{f_i} h) f_j\). Now define the observability Lie algebra, \(\mathcal{O}\), which is the span of the output function, \(h\), and all its Lie derivatives with respect to the control vector fields, \(f_i\):

\[
\mathcal{O} = \text{span}\{L_{X_1} h, L_{X_2} h, \ldots, L_{X_k} h\} \quad k = 0, 1, 2, \ldots
\tag{4}
\]

where \(X_i \in \{f_1, \ldots, f_m\}\), for \(i \in \{1, 2, \ldots, k\}\). If the Jacobian of the observability Lie algebra, \(d\mathcal{O}\), has \(\text{rank}(d\mathcal{O}) = n\) (or \(\text{rank}(d\mathcal{O}(x_0)) = n\)), then system (1) is said to be observable (locally at \(x_0\)). This check for observability is known in the literature as the observability rank condition. Note that the terms in \(\mathcal{O}\) include Lie derivatives with respect to the control vector fields, which indicates that the actuation and sensing can be coupled in nonlinear systems. We will exploit this coupling to shape the observability characteristics of the system.

In this paper, we focus on first-order nonholonomic systems in canonical form,

\[
x = u
\]

\[
z_{ij} = x_i u_j - x_j u_i \quad \forall \ i, j \in \{1, 2, \ldots, m\}, \ i > j
\tag{5}
\]

where \(x \in \mathbb{R}^m\) are the holonomic states, \(z \in \mathbb{R}^{m(m-1)/2}\) are the nonholonomic states, and \(u \in \mathbb{R}^m\) are the controls. This type of system was studied in the context of geometric and optimal control by Brockett [10] and later by Murray and Sastry [11].

The class of systems described by (5) are first-order Lie bracket controllable, meaning that the control Lie algebra constructed by the control vector fields and their first-order Lie brackets is full rank. Because the Lie brackets are necessary to guarantee full rank of the control Lie algebra, switching or area-generating controls must be used to steer the system from an initial configuration to an arbitrary final configuration.

These systems also have interesting observability characteristics. Consider the case when system (5) has an output function \(y = z\). Taking Lie derivatives of the output, we see that \(L_{x_i} z_{ij} = -x_j\) and \(L_{x_j} z_{ij} = x_i\), therefore \(x_i\) and \(x_j\) are observable with measurements of \(z_{ij}\) and use of the controls \(u_i\) and \(u_j\). We will refer to systems with this property as first Lie derivative observable. This result leads to the following theorems.

**Theorem 1 (Canonical Form Observability):** First-order nonholonomic systems in canonical form (5) are observable if and only if measurements of the nonholonomic states \(z\) are available.

**Proof:** If all nonholonomic states are measured, then the observability Lie algebra is \(\mathcal{O} = \text{span}\{z_{ij}, -x_j, x_i\} \forall \ i, j \in \{1, 2, \ldots, m\}, \ i > j\), and \(\text{rank}(d\mathcal{O}) = m(m + 1)/2 = \dim(x) + \dim(z)\), therefore the system is observable if all \(z\) states are measured and control actuation is utilized. Now, assume that one nonholonomic state \(z_{kl}\) is not measured. Since \(z_{kl}\) does not appear in the control vector fields, it will not be observable, and \(\text{rank}(d\mathcal{O}) \leq m(m + 1)/2 + 1\), thus the system will be unobservable.

**Theorem 2 (Actuation Required for Observability):**

System (5) with output \(y = z\) is observable if and only if actuation in at least two control channels is utilized.

**Proof:** The Lie derivative of one output with respect to a control vector field is

\[
L_{f_k} z_{ij} = \begin{cases} 
-x_j & k = i \\
x_i & k = j \\
0 & \text{otherwise}
\end{cases}
\]

Since we have \(z_{ij} \forall \ i, j \in \{1, 2, \ldots, m\}, i > j\), the span of the Lie derivative of all outputs with respect to a control vector field is \(\text{span}\{L_{f_k} z\} = \text{span}\{x_j\} \forall j \neq i\), which spans all \(x\) states except \(x_i\). Therefore, any two control vector fields \(f_k\) and \(f_{k'}\), \(k \neq k'\), have Lie derivatives that span all \(x\) states, and the system is observable with a minimum of two control inputs.

**Corollary 1:** If all \(z\) states are measured and one \(x\) state is measured, then actuation in \(u_i\) will give an observable system.

### III. Observability Optimization

The analysis of the previous section provides the control actuation channels that satisfy necessary and sufficient conditions for observability, however, it does not provide a metric for determining what control patterns provide the best observability. In this section, we will develop and solve an optimization problem for observability-optimal controls.

**A. Observability Cost Functional**

First, we must select an observability metric that will serve as our cost functional. Previous work by Krener and Ide [6] discussed the properties of the observability gramian of linear time varying systems and how they relate to local observability for nonlinear systems. To illustrate some of the results from [6], consider a nonlinear system, linearized about a nominal trajectory to obtain the linear time-varying system

\[
\dot{x}(t) = A(t, x^0(t), u^0(t)) x(t) + B(t) \dot{u}(t)
\]

\[
\dot{y}(t) = C(t, x^0(t)) x(t),
\tag{6}
\]

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where $x^0(t)$ and $u^0(t)$ are the nominal trajectory and corresponding controls about which the linearization is performed, \( \tilde{x}(t) = x(t) - x^0(t) \) and \( \tilde{u}(t) = u(t) - u^0(t) \) are the deviations from the nominal trajectory, and \( \tilde{y}(t) \) is the measurement deviation. Computing the output energy gives

\[
\|\tilde{y}(t)\| = \tilde{x}_0^T \left( \int_{t_0}^{t_f} \Phi(t_0, t)^T C(t)^T C(t) \Phi(t_0, t) dt \right) \tilde{x}_0, \tag{7}
\]

where \( \Phi(t_0, t) \) is the state transition matrix, \( \tilde{x}_0 \) is the initial state at time \( t_0 \), and \( W(t_0, t_f) \) is the observability gramian. The eigenvalues of \( W(t_0, t_f) \) give an indication of relative levels of observability between all possible initial conditions. The least observable (minimum output energy) initial condition direction corresponds to the minimum eigenvector of \( W \), and the most observable (maximum output energy) initial condition direction corresponds to the maximum eigenvector of \( W \). The inverse of the minimum eigenvalue of \( W \), \( \lambda_{\text{min}}^{-1}(W) \), is denoted the unobservability index in [6], which is a measure of the unobservability of the least observable mode.

Another important property of the observability gramian is the condition number, \( \kappa(W) = \lambda_{\text{max}}(W)/\lambda_{\text{min}}(W) \). We can visualize the condition number graphically by looking at the level sets of the output energy function, \( \|y(t)\| \). Fig. 1 shows ellipsoid level sets of the output energy for two example observability gramians. The radii of these ellipsoids are equal to the inverse of the eigenvalues of \( W \). The maximum eigenvector, \( \nu_1 \), is the direction with the largest gain, therefore a small perturbation in that direction yields an output energy equivalent to that of a larger perturbation in the direction of the smallest eigenvector. Therefore, an observability gramian with a large condition number indicates that the output energy is dominated by some modes, while others are difficult to observe.

The observability gramian also has statistical significance in the estimation problem. Consider a set of discrete measurements of system (6) corrupted by noise

\[
y_k = C(k\Delta t)x(k\Delta t) + v_k, \tag{8}
\]

where \( v \in \mathbb{R}^p \) is a vector of zero mean Gaussian random variables with covariance \( E[vv^T] = R \). Using a set of discrete measurements for \( k = 1, \ldots, N \), we can construct an estimate of the initial condition \( x_0 \). Stacking all the measurements together and noting that \( x(k\Delta t) = \Phi(t_0, k\Delta t)x_0 = \Phi_kx_0 \) for \( u = 0 \), we get a linear measurement model

\[
Y = \begin{bmatrix} C_1\Phi_1 \\ C_2\Phi_2 \\ \vdots \\ C_N\Phi_N \end{bmatrix} \tilde{x}_0 = Hx_0. \tag{9}
\]

Using a weighted least squares formulation, the minimum variance estimate is given by [12]

\[
\hat{x}_0 = (H^T R^{-1} H)^{-1} H^T R^{-1} Y. \tag{10}
\]

Computing the estimate covariance gives

\[
P = E[(\hat{x}_0^* - x_0^*)(\hat{x}_0^* - x_0^*)^T] = (H^T R^{-1} H)^{-1}. \tag{11}
\]

If the measurement noise variables are independent and identically distributed, then \( R = \sigma I \), where \( \sigma \) is the variance of each measurement, and the covariance simplifies to \( P = \sigma(H^T H)^{-1} \). Expanding the covariance matrix, we see that \( H^T H = \sum_k \Phi_k^T C_k^T C_k \Phi_k = W(1, N) \), which is the discrete time observability gramian. Therefore, the observability gramian is precisely the inverse of the estimate covariance scaled by the measurement noise variance. If the observability gramian eigenvalues can be manipulated (or even specified), then the estimation covariance is directly manipulated as a result. In this work, we consider the problem of maximizing the minimum eigenvalue of \( W \) as the primary cost functional, which corresponds to minimizing the maximum eigenvalue of \( P \) (i.e. minimizing the largest estimation covariance):

\[
\max_{x^0(t), u^0(t)} \lambda_{\text{min}}(W(t_0, t_f)) 
\]

subject to \( x^0 = \sum_{i=1}^{m} f_i(x^0(t))u_i \)

\( \dot{\Phi} = A(t, x^0(t), u^0(t))\Phi(t) \)

\( \dot{W} = \Phi^T(t)C^T(t, x^0(t))C(t, x^0(t))\Phi(t) \)

\( x^0(0) = x_0, \quad \Phi(0) = I, \quad W(0) = 0. \tag{12} \)

**B. Control for Optimal Observability**

The first-order nonholonomic system in canonical form (5) can be rewritten as a bilinear system without loss of generality to facilitate the observability optimization process:

\[
\dot{x} = u, \quad \dot{z} = F(u)x, \tag{13} \quad y = z.
\]

The matrix \( F(u) \in \mathbb{R}^{m(m-1)/2 \times m} \) consists of \( m-1 \) blocks, \( F(u) = [F_1(u)^T \ F_2(u)^T \ \ldots \ F_{m-1}(u)^T]^T \).
where each block \( F_i(u) \in \mathbb{R}^{(m-i) \times m} \) is given by
\[
F_i(u) = \begin{bmatrix}
0_{(i-1)} & u_{i+1} - u_i & 0 & \ldots & 0 \\
0_{(i-1)} & u_{i+2} & -u_i & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0_{(i-1)} & u_m & 0 & \ldots & -u_i
\end{bmatrix}, \tag{14}
\]
and \( 0_{(i-1)} \) is a row vector of zeros of length \((i - 1)\). As an example, when the length of \( u \) is four, \( F(u) \) is given by
\[
F(u) = \begin{bmatrix}
u_2 & -u_1 & 0 & 0 \\
u_3 & 0 & -u_1 & 0 \\
u_4 & 0 & 0 & -u_1 \\
0 & u_3 & -u_2 & 0 \\
u_4 & 0 & -u_2 & 0 \\
0 & 0 & u_4 & -u_3
\end{bmatrix}. \tag{15}
\]

Since the nonholonomic states \( z \) are directly measured, we are interested in optimizing the observability gramian associated with the holonomic states \( x \). Integrating to determine the output at time \( t \) gives
\[
y(t) = z_0 + \int_0^t F(u(\tau))x(\tau)d\tau = z_0 + \int_0^t F(u(\tau))\left(x_0 + \int_0^{\tau_1} x(\tau_2)d\tau_2\right)d\tau_1. \tag{16}
\]

Now, collect the terms influenced by the unknown initial condition of the holonomic states, and define a new output
\[
\eta = y - \left[z_0 + \int_0^t F(u(\tau))\left(\int_0^{\tau_1} x(\tau_2)d\tau_2\right)d\tau_1\right] = \int_0^t F(u(\tau_1))d\tau_1 x_0 = F\left(\int_0^t u(\tau_1)d\tau_1\right)x_0. \tag{17}
\]

For notational simplicity, define \( \xi(t) = \int_0^t u(\tau_1)d\tau_1 \). The output energy associated with holonomic states is now
\[
\|\eta(t)\| = x_0^T \int_0^{t_f} F(\xi(t))^TF(\xi(t))d\tau x_0 = x_0^T W_x(0, t_f)x_0, \tag{18}
\]
where \( W_x(0, t_f) \) is the observability gramian associated with the holonomic states. Expanding the terms inside the integral gives
\[
(F(\xi)^TF(\xi))_{ij} = \begin{cases}
-\xi_i \xi_j & i \neq j \\
\sum_{l \neq i} \xi_l^2 & i = j
\end{cases}. \tag{19}
\]

Based on the form of (19), we arrive at the following observability-optimal trajectory.

**Theorem 3 (Optimal Trajectory):** Cyclic controls of the form
\[
u_i^*(t) = \frac{2\pi j A_i}{t_f} \cos(2\pi j t / t_f),
\]
and corresponding state trajectories
\[
\xi_j^*(t) = A_j \sin(2\pi j t / t_f) \quad \forall \ j \in \{1, 2, \ldots, m\}
\]
optimize the observability gramian eigenvalues for first order nonholonomic systems in canonical form (5) with output \( y = z \).

**Proof:** First, note that \( \int_0^{t_f} \xi_i^*(t)\xi_j^*(t)dt = 0 \) and \( \int_0^{t_f} \xi_i^*(t)^2dt = A_i^2 t_f / 2 \). Substituting the optimal trajectory into (18), the \( ij \) entry of the observability gramian is
\[
(W_x(0, t_f))_{ij} = \begin{cases}
0 & i \neq j \\
\sum_{l \neq i} A_l^2 t_f / 2 & i = j
\end{cases}.
\]

Therefore, the observability gramian is diagonalized, and the eigenvalues of \( W_x(0, t_f) \) can be placed using appropriate amplitudes \( A_j \).

**Remark 1:** Note that the choice of integer \( j \) in the sinusoidal frequency is arbitrary. We use \( j \) as the integer frequency multiplier for state \( \xi_i \) for convenience only, but any integer can be used such that no \( \xi_i \) and \( \xi_j \) have the same frequency.

The desired observability gramian eigenvalues form a set of equations that can be solved for the optimal state trajectory amplitudes
\[
\frac{t_f}{2} (11^T - I) A^2 = \lambda, \tag{20}
\]
where \( 1 \) is a column vector of ones, \( A^2 \) is a column vector of the amplitudes squared, and \( \lambda \) is a column vector of desired observability gramian eigenvalues with \( \lambda_i \) representing the eigenvalue associated with the \( x_i \) state. Showing that \( (11^T - I) \) is a full-rank matrix is straightforward, therefore the solution for \( A^2 \) always exists and is unique. We will now solve for \( A^2 \) explicitly. Subtracting the first equation of (20) from the remaining equations, we have
\[
A_i^2 = A_1^2 + \frac{2}{t_f} (\lambda_1 - \lambda_i) \quad \forall \ i \in \{2, 3, \ldots, m\}. \tag{21}
\]

Substituting these expressions back into the first equation and solving for \( A_1^2 \) and \( A_i^2 \) gives
\[
A_i^2 = \frac{2}{t_f} \left(-\lambda_i + \frac{1}{(m-1)} \sum_{j=1}^{m} \lambda_j\right) \quad \forall \ i. \tag{22}
\]
For this solution to be feasible, we must have \( A_i^2 \geq 0 \), thus
\[
0 < \lambda_i \leq \frac{1}{(m-2)} \sum_{j \neq i} \lambda_j \tag{23}
\]
for a feasible solution. The constraint imposed by (23) essentially limits the spread of the eigenvalues of \( W_x(0, t_f) \). The case when all \( \lambda_i \) are equivalent (i.e. a condition number of \( \kappa(W_x(0, t_f)) = 1 \)) is always a feasible. The feasible set of eigenvalues is the interior of a cone in the positive orthant with the vertex at the origin. When \( m = 2 \), the constraint (23) does not limit our choice of \( \lambda \), and the feasible set is the entire positive orthant. When \( m > 2 \), the feasible space is enclosed by \( m \) hyper-planes passing through the origin with equations \( -\lambda_i + 1/(m-2) \sum_{j \neq i} \lambda_j = 0 \). As \( m \) increases,
the cone becomes smaller, and in the limit $m \to \infty$, the cone reduces to the line of all $\lambda_i$ equivalent. Practically speaking, the constraint (23) does not impose undesirable limitations on our choice of $\lambda$ since a condition number of 1 is desirable and always feasible.

C. Example Calculation

We will now demonstrate calculation of the optimal trajectory for the case when $m = 3$. Suppose that we desire observability gramian eigenvalues of $\lambda = [50 \ 50 \ 50]^T$ over a time horizon of $t_f = 2$. Then using (22), we have state trajectory amplitudes of

$$A_i = \sqrt{-50 + \frac{1}{2}(150)} = 5,$$

(24)

which results in an observability gramian of $W_x(0, 2) = \text{diag} \{ [50 \ 50 \ 50] \}$. Note that the negative square root could also be chosen for the amplitude $A_i$ (i.e. -5) without loss of optimality. The resulting state trajectory would simply be mirrored in the $i$th axis direction. The optimal trajectory traces a figure-8 in the state space, as shown in Fig. 2. It is shown here with the origin of the trajectory at the origin of the state space, but the optimal trajectory is translationally invariant, so the initial position is arbitrary.

If we are most concerned with estimates of $x_1$, then the first eigenvalue of $W_x(0, t_f)$ can be increased to e.g. $\lambda = [75 \ 50 \ 50]^T$ which yields state amplitudes of $A = [\sqrt{12.5} \ \sqrt{37.5} \ \sqrt{37.5}]$. To reduce the covariance of an estimate of state $x_i$, the control amplitude on $u_i$ is decreased while the amplitudes on all other control channels are increased. This behavior is intuitive, since the output energy due to $x_i$ is dependent on all control inputs except $u_i$.

IV. SIMULATION RESULTS

The first-order nonholonomic system (5) and an accompanying state estimator are simulated to demonstrate the usefulness of the observability optimization. Using a linearization about the nominal trajectory, this system is represented as a linear time-varying system, and a Kalman Filter is used to find a minimum variance state estimate. The system model is represented by

$$\frac{d}{dt} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F(u^0) & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} I \\ -F(x^0) \end{bmatrix} \bar{u} + w,$$

(25)

where $w$ and $v$ are the zero-mean Gaussian process and measurement noises, respectively, with covariances $E[ww^T] = Q$ and $E[vv^T] = R$.

Three simulation scenarios are presented here for the example system with $m = 3$ controls. The first scenario uses the optimal nominal control input defined by Theorem 3 with observability gramian eigenvalues placed at $\lambda = [50 \ 50 \ 50]^T$ over a time horizon of $t_f = 2$. The second scenario uses a suboptimal nominal control defined by $u_j(t) = u_j^*(t) + jt$. The final scenario uses a nominal control input of $u_1(t) = u_1^*(t)$ and $u_j(t) = 0 \ \forall j \in \{2, 3, \ldots, m\}$, which results in an observable system as proven in Theorem 2.

A continuous-discrete Kalman Filter is implemented in simulation according to the formulation in [12] with process and measurement noises set to $Q = 1 \times 10^{-5}I$ and $R = 1 \times 10^{-2}I$. Estimates are initialized to $\hat{x}(0) = 0$ and $\hat{z}(0) = \tilde{z}(0)$ with an initial covariance matrix of $P = I$. True states are initialized to $x(0) = 10$ and $z(0) = 0$.

Simulation results for scenario 1 are shown in Figs. 3 and 4. State estimates for the unknown holonomic states converge quickly to the true state values with corresponding low covariances. If we define the estimate convergence time as $t_c = \min \{ t \mid |\hat{x}_i(\tau) - x_i(\tau)| \leq 0.1|x_i(\tau)|, \ \forall \tau \geq t, \ i \in \{1, 2, \ldots, m\} \}$ (i.e. the 10% settling time), then the optimal trajectory yields a convergence time of $t_c = 0.13$ seconds. The average estimate covariance after convergence for scenario 1 is computed to be $\tilde{\sigma} = 0.03$.

In contrast, results for the suboptimal scenario 2 shown in Figs. 5 and 6 show a significantly larger convergence time and average covariance. The suboptimal input yields a convergence time of $t_c = 0.60$ seconds, which is 4.5 times longer than the optimal trajectory, and an average covariance of $\bar{\sigma} = 0.09$, which is three times that of the optimal trajectory covariance.

Simulation results for the third scenario are shown in Figs. 7 and 8. As predicted by Theorem 2, the estimator does not converge to the true value of all states due to the use of only one control.

V. CONCLUSION

In this paper, we formulated and solved an optimal control problem to maximize the observability of first-order nonholonomic systems in canonical form. These types of systems are of interest due to fundamental difficulties imposed by the nonholonomic constraints. Furthermore, the system dynamics are diffeomorphic to dynamics common to many engineering problems, such as the unicycle model for robotic vehicle navigation.

Observability analysis of the system showed that all nonholonomic states $z_{ij}$ must be measured in the output and...
a minimum of two control channels must be utilized to yield an observable system. The system was shown to be first-order Lie derivative observable, meaning that first-order Lie derivatives with respect to the control vector fields are required for observability.

The properties of the observability gramian for linear time-varying systems were discussed as metrics for observability optimization. The link between observability gramian eigenvalues and estimator covariances was established, which motivated the placement of observability gramian eigenvalues as the primary control objective. Sinusoidal state trajectories at integer related frequencies were shown to be the observability-optimal trajectories for this system, which allows the eigenvalues of the observability gramian to be placed almost arbitrarily. In turn, the steady-state estimation covariances can be almost arbitrarily specified.

Simulation results demonstrated the power of observability optimization, showing that the optimal trajectory yielded 4.5 times faster convergence and three times lower estimation covariances than a suboptimal trajectory.

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