AA599: Geometric Methods for Nonlinear Control Systems

Midterm Exam

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The exam consists of four questions, worth a total of 60 points. The point values for each section are shown on the left. The exam is due in my office or my department mailbox (Gug. 206) by 5pm Friday May 7.

The exam is open book. You may use the textbook (Isidori), course handouts, lecture and class notes, course problem sets and solutions, and handwritten notes. No other books are allowed. You may use a computer or calculator only for carrying out numerical computations.

Problem 1

Consider the sphere in $\mathbb{R}^3$ defined by

$$S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$$

and the set in $\mathbb{R}^6$ defined by

$$T = \{(x, y, z, p_1, p_2, p_3) | x^2 + y^2 + z^2 = 1, xp_1 + yp_2 + zp_3 = 0\}$$

(a) (5 pts) Given $\phi_1 = x_1^2 + x_2^2 + x_3^2$ and $\phi_2 = x_1p_1 + x_2p_2 + x_3p_3$, evaluate the rank of the Jacobian

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial \mathbf{x}} \\ \frac{\partial \phi_2}{\partial \mathbf{x}} \end{bmatrix}$$

(b) (5 pts) Do the same for $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T p = 0$.

(c) (5 pts) Describe the geometry of T.

Problem 2

(10 pts) Consider the three vector fields on $\mathbb{R}^1$ defined by $\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}$, and $x^2\frac{\partial}{\partial x}$. What Lie algebra do they generate?

Problem 3

Consider the following model of a car (Figure 1). The configuration space is $M = \mathbb{R}^2 \times S^1 \times S^1$

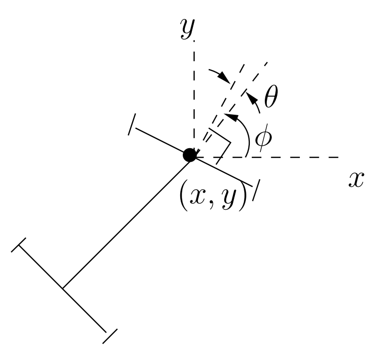

Figure 1: Model of a car.

parameterized by (x, y, ϕ, θ), where (x, y) are the Cartesian coordinates of the center of the front axis, the angle ϕ measures the direction in which the car is headed, and θ is the angle made by the front wheels with the car. (More realistically we take $-\theta_{max} < \theta < \theta_{max}$.)

There are two input vector fields, called Steer and Drive. Clearly Steer $= \frac{\partial}{\partial \theta}$, while after some analysis we see that, in the appropriate units, Drive$=\cos(\phi+\theta)\frac{\partial}{\partial x}+\sin(\phi+\theta)\frac{\partial}{\partial y}+\sin\theta\frac{\partial}{\partial \phi}$.

(a) (5 pts) Show that

$$[\text{Steer}, \text{Drive}] = -\sin(\phi+\theta)\frac{\partial}{\partial x} + \cos(\phi+\theta)\frac{\partial}{\partial y} + \cos(\theta)\frac{\partial}{\partial \phi} =: \text{Wriggle}.$$

(b) (5 pts) Define Slide=$-\sin(\phi)\frac{\partial}{\partial x} + \cos(\phi)\frac{\partial}{\partial y}$. Show that [Steer,Wriggle]=-Drive and [Wriggle,Drive]=Slide. Additionally compute the bracket of Slide with Steer, Drive and Wriggle.

(c) (5 pts) Show that this system is controllable.

(d) (5 pts) Given the observation $h(x, y, \psi, \theta) = x$, can the full system state be recovered? If so, state any necessary conditions.

Problem 4

Suppose that N and $\widetilde{N}$ are open sets in $\mathbb{R}^n$ and that $\psi : N \to \widetilde{N}$ is a smooth map with a smooth inverse $\psi^{-1} : \widetilde{N} \to N$. Suppose that

$$\dot{x} = f(x(t)),\ x(t) \in \mathbb{R}^n$$

(a) (5 pts) Find a differential equation for $z(t) = \psi(x(t))$.

(b) (5 pts) We adopt the notation

$$\dot{z}(t) = \hat{f}(z(t))$$

and also if

$$\dot{x} = g(x(t))$$

then

$$\dot{z}(t) = \hat{g}(z(t))$$

Show that if

$$\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g = h$$

then

$$\frac{\partial \hat{g}}{\partial z}\hat{f} - \frac{\partial \hat{f}}{\partial z}\hat{g} = \hat{h}$$

(c) (5 pts) Give a two sentence explanation of this fact.