

AA599: Geometric Methods for Nonlinear Control Systems

Midterm Exam

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The exam consists of four questions, worth a total of 60 points. The point values for each section are shown on the left. The exam is due in my office or my department mailbox (Gug. 206) by 5pm Friday May 7.

The exam is open book. You may use the textbook (Isidori), course handouts, lecture and class notes, course problem sets and solutions, and handwritten notes. No other books are allowed. You may use a computer or calculator only for carrying out numerical computations.

Problem 1

Consider the sphere in \mathbb{R}^3 defined by

$$S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$$

and the set in \mathbb{R}^6 defined by

$$T = \{(x_1, x_2, x_3, p_1, p_2, p_3) | x_1^2 + x_2^2 + x_3^2 = 1, x_1 p_1 + x_2 p_2 + x_3 p_3 = 0\}$$

with $\mathbf{x} = [x_1, x_2, x_3, p_1, p_2, p_3]$.

- (a) (5 pts) Given $\phi_1 = x_1^2 + x_2^2 + x_3^2$ and $\phi_2 = x_1 p_1 + x_2 p_2 + x_3 p_3$, evaluate the rank of the Jacobian

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial \mathbf{x}} \\ \frac{\partial \phi_2}{\partial \mathbf{x}} \end{bmatrix}$$

The Jacobian is given by

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial \mathbf{x}} \\ \frac{\partial \phi_2}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & x_1 & x_2 & x_3 \end{bmatrix}$$

Given that $x_1^2 + x_2^2 + x_3^2 = 1$, we know that the rank of the Jacobian must always be 2.

- (b) (5 pts) Do the same for $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}^T \mathbf{x} = 1$ and $\mathbf{x}^T \mathbf{p} = 0$.

As above, we have

$$J = \begin{bmatrix} [x_1, \dots, x_n] & 0 \\ [p_1, \dots, p_n] & [x_1, \dots, x_n] \end{bmatrix}$$

and the rank will always be 2.

- (c) (5 pts) Describe the geometry of T .

T is the tangent space to the sphere S^n .

Problem 2

(10 pts) Consider the three vector fields on \mathbb{R}^1 defined by $\frac{\partial}{\partial x}$, $x \frac{\partial}{\partial x}$, and $x^2 \frac{\partial}{\partial x}$. What Lie algebra do they generate?

To determine the algebra, we begin by taking Lie brackets of the three terms:

$$\begin{aligned} \left[\frac{\partial}{\partial x}, x \frac{\partial}{\partial x} \right] &= \frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x} \right) - x \frac{\partial}{\partial x} \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \\ \left[\frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} \right] &= \frac{\partial}{\partial x} \left(x^2 \frac{\partial}{\partial x} \right) - x^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} = 2x \frac{\partial}{\partial x} \\ \left[x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} \right] &= x \frac{\partial}{\partial x} \left(x^2 \frac{\partial}{\partial x} \right) - x^2 \frac{\partial}{\partial x} \left(x \frac{\partial}{\partial x} \right) = x^2 \frac{\partial}{\partial x} \end{aligned}$$

All of these terms are already included in the original set, so the Lie algebra is just the original set of vector fields. Note that these vector fields are not linearly independent as they are all of the form $\alpha(x) \frac{\partial}{\partial x}$.

Problem 3

Consider the following model of a car (Figure 1). The configuration space is

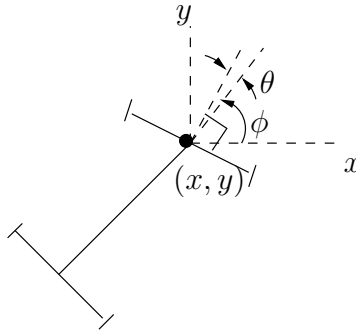


Figure 1: Model of a car.

$M = \mathbb{R}^2 \times S^1 \times S^1$ parameterized by (x, y, ϕ, θ) , where (x, y) are the Cartesian coordinates of the center of the front axis, the angle ϕ measures the direction in which the car is headed, and θ is the angle made by the front wheels with the car. (More realistically we take $-\theta_{max} < \theta < \theta_{max}$.)

There are two input vector fields, called Steer and Drive. Clearly $\text{Steer} = \frac{\partial}{\partial \theta}$, while after some analysis we see that, in the appropriate units, $\text{Drive} = \cos(\phi + \theta) \frac{\partial}{\partial x} + \sin(\phi + \theta) \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial \phi}$.

(a) (5 pts) Show that

$$[\text{Steer}, \text{Drive}] = -\sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\phi + \theta) \frac{\partial}{\partial y} + \cos(\theta) \frac{\partial}{\partial \phi} =: \text{Wriggle}.$$

$$\begin{aligned} [\text{Steer}, \text{Drive}] &= \left[\frac{\partial}{\partial \theta}, \cos(\phi + \theta) \frac{\partial}{\partial x} + \sin(\phi + \theta) \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial \phi} \right] \\ &= -\sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\phi + \theta) \frac{\partial}{\partial y} + \cos(\theta) \frac{\partial}{\partial \phi} \end{aligned}$$

(b) (5 pts) Define $\text{Slide} = -\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y}$. Show that $[\text{Steer}, \text{Wriggle}] = -\text{Drive}$ and $[\text{Wriggle}, \text{Drive}] = \text{Slide}$. Additionally compute the bracket of Slide with Steer, Drive and Wriggle.

$$\begin{aligned}
[\textit{Steer}, \textit{Wriggle}] &= \left[\frac{\partial}{\partial \theta}, -\sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\phi + \theta) \frac{\partial}{\partial y} + \cos(\theta) \frac{\partial}{\partial \phi} \right] \\
&= -\cos(\phi + \theta) \frac{\partial}{\partial x} - \sin(\phi + \theta) \frac{\partial}{\partial y} - \sin(\theta) \frac{\partial}{\partial \phi} \\
&= -\textit{Drive}
\end{aligned}$$

$$\begin{aligned}
[\textit{Wriggle}, \textit{Drive}] &= \left[-\sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\phi + \theta) \frac{\partial}{\partial y} + \cos(\theta) \frac{\partial}{\partial \phi}, \right. \\
&\quad \left. \cos(\phi + \theta) \frac{\partial}{\partial x} + \sin(\phi + \theta) \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial \phi} \right] \\
&= -\cos(\theta) \sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\theta) \cos(\phi + \theta) \frac{\partial}{\partial y} \\
&\quad + \sin(\theta) \cos(\phi + \theta) \frac{\partial}{\partial x} + \sin(\theta) \sin(\phi + \theta) \frac{\partial}{\partial y} \\
&= -\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y} \\
&= \textit{Slide}
\end{aligned}$$

$$\begin{aligned}
[\textit{Slide}, \textit{Steer}] &= \left[-\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta} \right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\textit{Slide}, \textit{Drive}] &= \left[-\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y}, \cos(\phi + \theta) \frac{\partial}{\partial x} + \sin(\phi + \theta) \frac{\partial}{\partial y} + \sin \theta \frac{\partial}{\partial \phi} \right] \\
&= -\sin(\theta) \cos(\phi) \frac{\partial}{\partial x} - \sin(\theta) \sin(\phi) \frac{\partial}{\partial y}
\end{aligned}$$

$$\begin{aligned}
[\textit{Slide}, \textit{Wriggle}] &= \left[-\sin(\phi) \frac{\partial}{\partial x} + \cos(\phi) \frac{\partial}{\partial y}, -\sin(\phi + \theta) \frac{\partial}{\partial x} + \cos(\phi + \theta) \frac{\partial}{\partial y} + \cos(\theta) \frac{\partial}{\partial \phi} \right] \\
&= -\cos(\theta) \cos(\phi) \frac{\partial}{\partial x} - \cos(\theta) \sin(\phi) \frac{\partial}{\partial y}
\end{aligned}$$

(c) (5 pts) Show that this system is controllable.

As the system has no drift, we simply need to check that the vector fields from the previous part span the tangent space at all points. The vector field $\frac{\partial}{\partial \theta}$ is given, so that direction is always spanned. The directions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are spanned everywhere with the combination of $[\textit{Steer}, \textit{Drive}]$, $[\textit{Slide}, \textit{Drive}]$, and $[\textit{Slide}, \textit{Wriggle}]$. The direction $\frac{\partial}{\partial \phi}$ is then given by the use of the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in combination with $\textit{Wriggle}$ and \textit{Drive} .

(d) (5 pts) Given the observation $h(x, y, \psi, \theta) = x$, can the full system state be recovered? If so, state any necessary conditions.

The observability algebra is given by the output and the Lie derivative of the output with respect to the vector fields above.

$$\mathcal{O} = [x, \cos(\phi + \theta), -\sin(\phi + \theta), -\sin(\phi), -\sin(\theta) \cos(\phi), -\cos(\theta) \cos(\phi)]$$

Computing $d\mathcal{O}$ we have

$$d\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\cos(\phi + \theta) & -\cos(\phi + \theta) \\ 0 & 0 & -\cos(\phi) & 0 \\ 0 & 0 & \sin(\theta) \sin(\phi) & -\cos(\theta) \cos(\phi) \\ 0 & 0 & \cos(\theta) \sin(\phi) & \sin(\theta) \cos(\phi) \end{bmatrix}$$

This matrix can never have full rank, so the system is not observable.

Problem 4

Suppose that N and \tilde{N} are open sets in \mathbb{R}^n and that $\psi : N \rightarrow \tilde{N}$ is a smooth map with a smooth inverse $\psi^{-1} : \tilde{N} \rightarrow N$. Suppose that

$$\dot{x} = f(x(t)), \quad x(t) \in \mathbb{R}^n$$

(a) (5 pts) Find a differential equation for $z(t) = \psi(x(t))$.

The inverse of the relationship is $x(t) = \psi^{-1}(z(t))$.

$$\begin{aligned} \dot{z} &= \left. \frac{\partial \psi}{\partial x} \right|_{x=\psi^{-1}(z)} \dot{x} \\ &= \left. \frac{\partial \psi}{\partial x} \right|_{x=\psi^{-1}(z)} f(x(t)) \\ &= \frac{\partial \psi(\psi^{-1}(z))}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \\ &= \frac{\partial z}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \end{aligned}$$

(b) (5 pts) We adopt the notation

$$\dot{z}(t) = \hat{f}(z(t))$$

and also if

$$\dot{x} = g(x(t))$$

then

$$\dot{z}(t) = \hat{g}(z(t))$$

Show that if

$$\frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g = h$$

then

$$\frac{\partial \hat{g}}{\partial z}\hat{f} - \frac{\partial \hat{f}}{\partial z}\hat{g} = \hat{h}$$

$$\begin{aligned} [\hat{f}, \hat{g}] &= \frac{\partial \hat{g}}{\partial z}\hat{f} - \frac{\partial \hat{f}}{\partial z}\hat{g} \\ &= \frac{\partial}{\partial z} \left(\frac{\partial z}{\partial \psi^{-1}(z)} g(\psi^{-1}(z)) \right) \hat{f}(z) - \frac{\partial}{\partial z} \left(\gamma \frac{\partial z}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \right) \hat{g}(z) \\ &= \frac{\partial^2 z}{\partial z \partial \psi^{-1}(z)} g(\psi^{-1}(z)) \frac{\partial z}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \\ &\quad + \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial g}{\partial \psi^{-1}(z)} \frac{\partial \psi^{-1}(z)}{\partial z} \frac{\partial z}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \\ &\quad - \frac{\partial^2 z}{\partial z \partial \psi^{-1}(z)} f(\psi^{-1}(z)) \frac{\partial z}{\partial \psi^{-1}(z)} g(\psi^{-1}(z)) \\ &\quad - \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial f}{\partial \psi^{-1}(z)} \frac{\partial \psi^{-1}(z)}{\partial z} \frac{\partial z}{\partial \psi^{-1}(z)} g(\psi^{-1}(z)) \\ &= \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial g}{\partial \psi^{-1}(z)} \frac{\partial \psi^{-1}(z)}{\partial z} \frac{\partial z}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) \\ &\quad - \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial f}{\partial \psi^{-1}(z)} \frac{\partial \psi^{-1}(z)}{\partial z} \frac{\partial z}{\partial \psi^{-1}(z)} g(\psi^{-1}(z)) \\ &= \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial g}{\partial \psi^{-1}(z)} f(\psi^{-1}(z)) - \frac{\partial z}{\partial \psi^{-1}(z)} \frac{\partial f}{\partial \psi^{-1}(z)} g(\psi^{-1}(z)) \\ &= \frac{\partial z}{\partial \psi^{-1}(z)} [f(\psi^{-1}(z)), g(\psi^{-1}(z))] \end{aligned}$$

(c) (5 pts) Give a two sentence explanation of this fact.

The structure of the vector fields is coordinate free.