AA599: Geometric Methods for Nonlinear Control Systems
Lecture Notes
Trajectory Tracking

Kristi A. Morgansen
Department of Aeronautics and Astronautics

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We will base our construction of controllers for trajectory tracking on solutions to the optimal control problem for the nonholonomic integrator. As we will show later, constructing controllers for the nonholonomic integrator is the key building block for general nonlinear system controllers. The reason is that the nonholonomic integrator can be used as a basis structure for nonlinear systems that cannot be feedback linearized.

To begin, recall that the nonholonomic integrator is given by

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1\end{aligned}$$

The solution to the optimization problem

$$\min \int_0^1 (u_1^2 + u_2^2)dt$$

subject to fixed endpoints can be found either with the Euler-Lagrange equations or the Maximum Principle. the end result is solutions of the form

$$\begin{aligned}x_1(t) &= \alpha_1 + \alpha_3\left(1 - \cos(\mu t)\right) - \alpha_4 \sin(\mu t)\\ x_2(t) &= \alpha_2 - \alpha_4\left(1 - \cos(\mu t)\right) - \alpha_3 \sin(\mu t).\end{aligned}$$

where the coefficients are found by applying the boundary conditions (see the optimization notes).

Now consider the more general class of nonlinear systems

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

and the minimization problem

$$\min \frac{1}{2}\int_0^1 u^T u dt$$

The easiest case is found by assuming that $m = n$ and that the vector fields f_i are independent in a neighborhood of $x = 0$. We can then write

$$\dot{x} = G(x)u \Rightarrow u = G^{-1}(x)\dot{x}$$

so that we have

$$\min \frac{1}{2}\int_0^1 u^T u dt = \frac{1}{2}\int_0^1 \dot{x}^T G^{-1^T}(x) G^{-1}(x)\dot{x} dt = \frac{1}{2}\int_0^1 \dot{x}^T \widetilde{G}(x)\dot{x} dt$$

where $\widetilde{G}(x)$ is referred to as the system metric. Writing out the Euler-Lagrange equations for this system, we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \Rightarrow \tilde{G}\ddot{x} + \left(\frac{\partial}{\partial x}\widetilde{G}(x)\right)(\dot{x},\dot{x}) & \end{aligned}$$

where $\widetilde{G}(x)$ is an $n \times n$ matrix, $\frac{\partial \widetilde{G}}{\partial x}$ is an $n \times n \times n$ tensor, and $\frac{\partial \widetilde{G}}{\partial x}(\dot{x},\dot{x})$ means "evaluate the 3-tensor in the direction $(\dot{x},\dot{x})$. Specifically

$$\begin{aligned} \ddot{x} + \widetilde{G}^{-1}(x)\left(\frac{\partial}{\partial x}\widetilde{G}(x)\right)(\dot{x},\dot{x}) &= 0 \\ \ddot{x}_i + \sum_{j,k}\gamma_{ijk}\dot{x}_j\dot{x}_k &= 0 \end{aligned}$$

where γ_{ijk} are the Christoffel symbols. For a mechanical system with no potential forces and no external forces, $\widetilde{G}$ is the mass matrix and the Christoffel symbols are the combined Coriolis and Centrifugal force terms. These equations are the geodesic equations for the metric $\widetilde{G}$. Therefore this optimal control problem is equivalent to finding the shortest path between points on our manifold subject to the system dynamics.

We will have different behavior if we have fewer vector fields than states, but the system is controllable. In particular, for the nonholonomic integrator, note that

$$\begin{aligned} \dot{x}_3 &= x_1\dot{x}_2 - x_2\dot{x}_1 \\ \Rightarrow x_3(t) &= \int_0^t x_1\dot{x}_2 - x_2\dot{x}_1 dt \\ &= \int_0^{x_2} x_1 dx_2 - \int_0^{x_1} x_2 dx_1 \end{aligned}$$

which gives the signed area of the curve formed by $x_1 - x_2$. If we do not want to advance any distance in the x_3 direction, we would then use a straight line in $x_1 - x_2$ space. The solution to our minimization problem, to find the shortest arc length with the largest area, is given by arcs of circles. For a full circle (no net motion in the $x_1 - x_2$ plane), we have an area of πr^2 and an arc length of $2\pi r$. Since $x_{3,f} = \pi r^2$ we have $r = \sqrt{\frac{x_{3,f}}{\pi}}$ and the arc length of our curve is $4\pi x_{3,f}$. Note that motion in the x_3 direction is homogeneous of degree 1 in x_1, x_2, but motion in the $x_1 - x_2$ plane is homogeneous of degree 2. Therefore it costs less to go a long distance than in the $x_1 - x_2$ plane but more to go a short distance. So u_1 and u_2 have more effect on x_3 further from the origin. Clearly then it is better to use one large circle to move between two points in the x_3 direction rather than several small circles.

These results show us how to get from one point to another in a certain optimal sense. But what about following a trajectory? Consider the case of linear systems:

$$\dot{x} = Ax + Bu, \ y = Cx$$

Given that $y(\cdot) : [0, T] \to T\mathbb{R}^p$, can we find $u : [0, T] \to \mathbb{R}^m$ such that u forces y to follow a desired path? i.e. can we find an inverse to the system? Well for A, B LTI we can directly integrate to get

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\sigma)}Bu(\sigma)d\sigma, \ y(0) = Cx(0)$$

Can we then find initial conditions $x(0)$ and u such that $y(t)$ is a desired value for $0 \leq t \leq T$. For a linear system we have

$$(y(t) - Ce^{At}x(0)) = \int_0^t Ce^A(t-\sigma)Bu(\sigma)d\sigma$$

Taking the Laplace transform of this equation, we have

$$\begin{aligned} \tilde{y}(x) - C(Is - A)^{-1}x(0) &= C(sI - A)^{-1}B\tilde{u}(s) \\ \tilde{u}(s) &= \left(C(sI - A)^{-1}B\right)^{\#}(y(s) - C(sI - A)^{-1}x(0)) \\ \tilde{u}(s) &= G(s)(y(s) - C(sI - A)^{-1}x(0)) \end{aligned}$$

If $G(s)$ has rank equal to the dimension of y, then we can find for each suitable $\tilde{y}$ a $\tilde{u}$ that makes the system produce the desired y. This result tells us something about the invertibility of linear systems. We would like to know what can be said about nonlinear systems. We will address this question with both exact and approximate tracking.

For exact tracking consider the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \ y = h(x)$$

from which we have

$$\dot{y} = \frac{\partial h}{\partial x}f_0(x) + \sum_{i=1}^{m}\frac{\partial h}{\partial x}f_i(x)u_i$$

We would like to know the conditions for which we can find u when we are given $y(\cdot)$. Continuing with this problem we have

$$y = h(x)$$

$$
\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial x} f_0(x) + \sum_{i=1}^{m} \frac{\partial h}{\partial x} f_i(x) u_i \\
\ddot{y} &= (\cdot)\dot{u} \\
y^{(3)} &= (\cdot)\ddot{u}
\end{aligned}
$$

this problem looks to be difficult in general, so we may not be able to follow a trajectory exactly, but perhaps we can get close with some choice of u.

For example, consider the system

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 - x_2 u_1
\end{aligned}
$$

and the task of tracking the desired trajectories $x_{1,d}(t)$, $x_{2,d}(t)$, $x_{3,d}(t)$. Now consider the inputs (motivated by the optimal control solution)

$$
\begin{aligned}
u_1(t) &= u_{10}(t) + \alpha_1(t)\sin(\omega_1 t) \\
u_2(t) &= u_{20}(t) + \alpha_2(t)\cos(\omega_2 t)
\end{aligned}
$$

We then have

$$
\begin{aligned}
x_1(t) &= x_1(0) + \int_0^t (u_{10}(\sigma) + \alpha_1(\sigma)\sin(\omega_1\sigma))d\sigma \\
x_2(t) &= x_2(0) + \int_0^t (u_{20}(\sigma) + \alpha_2(\sigma)\cos(\omega_2\sigma))d\sigma \\
x_3(t) &= x_3(0) + \int_0^t \left[x_1(0) + \int_0^\tau (u_{10}(\sigma) + \alpha_1(\sigma)\sin(\omega_1\sigma))d\sigma\right]\left[u_2(\tau) + \alpha_2(\tau)\cos(\omega_2\tau)\right] d\tau \\
&\quad - \int_0^t \left[x_2(0) + \int_0^\tau (u_{20}(\sigma) + \alpha_2(\sigma)\cos(\omega_2\sigma))d\sigma\right]\left[u_1(\tau) + \alpha_1(\tau)\cos(\omega_1\tau)\right] d\tau
\end{aligned}
$$

In order to evaluate the integrals we will use the Riemann-Lebesgue Lemma:

Lemma 0.1 *Let f be of bounded variation on $[a,b]$ and let $\phi \in [0, 2\pi]$. Then*

$$
\lim_{\omega\to\infty} \int_a^b f(t)\cos(\omega t + \phi)dt = o(\frac{1}{\omega})
$$

So as ω becomes large $o(\frac{1}{\omega})$ becomes small. How does this help us? Consider the term $x_1(t)$:

$$
x_1(t) = x_1(0) + \int_0^t u_{10}(\sigma)d\sigma + \int_0^t \alpha_1(\sigma)\sin(\omega_1\sigma)d\sigma
$$

The last term is then approximately $o(\frac{1}{\omega_1})$ (where $\phi = -\pi/2$). We then have

$$
x_1(t) \approx x_1(0) + \int_0^t u_{10}(\sigma)d\sigma
$$

We will assume that ω will be large compared with the highest frequency present in $x_{1d}(t)$, $x_{2d}(t)$ and $x_{3d}(t)$. These results are similar to averaging but slightly different in application.

$$\begin{aligned}
x_1(t) &= x_1(0) + \int_0^t u_{10}(\sigma)d\sigma + \int_0^t \alpha_1(\sigma)\sin(\omega_1\sigma)d\sigma \\
&= x_1(0) + \int_0^t u_{10}(\sigma)d\sigma - \alpha_1(\sigma)\frac{1}{\omega_1}\cos(\omega_1\sigma)\Big|_0^t + \int_0^t \dot{\alpha}_1(\sigma)\cos(\omega_1\sigma)d\sigma \\
&= x_1(0) + \int_0^t u_{10}(\sigma)d\sigma - \frac{1}{\omega_1}\left(\alpha_1(t)\cos(\omega_1 t) - \alpha_1(0)\right) + \int_0^t \alpha_1(\sigma)\cos(\omega_1\sigma)d\sigma \\
&= x_1(0) + \int_0^t u_{10}(\sigma)d\sigma + \underbrace{\frac{1}{\omega_1}size(\alpha_1) + size\dot{\alpha}_1}_{\text{small if } \omega_1 \text{ large and } \dot{\alpha}_1 \text{ small}}
\end{aligned}$$

Similarly $x_2(t) = x_2(0) + \int_0^t u_{20}(\sigma)d\sigma + \frac{1}{\omega_2}(\cdot) + \int_0^t \dot{\alpha}_2$.

Suppose $x_1(0) = x_2(0) = x_3(0) = 0$. Then

$$\int_0^t u_{10}(\sigma)d\sigma = \mathcal{U}_1(t), \quad \int_0^t u_{20}(\sigma)d\sigma = \mathcal{U}_2(t)$$

and

$$\begin{aligned}
x_3(t) &= \int_0^t \left(\mathcal{U}_1(\sigma) + \int_0^\sigma \alpha_1(\eta)\sin(\omega_1\eta)d\eta\right)\cdot\left(u_{20}(\sigma) + \alpha_2(\sigma)\cos(\omega_2\sigma)\right)d\sigma \\
&\qquad - \int_0^t \left(\mathcal{U}_2(\sigma) + \int_0^\sigma \alpha_2(\eta)\cos(\omega_2\eta)d\eta\right)\cdot\left(u_{10}(\sigma) + \alpha_1(\sigma)\sin(\omega_1\sigma)\right)d\sigma \\
&= \int_0^t \mathcal{U}_1(\sigma)u_{20}(\sigma) - \mathcal{U}_2(\sigma)u_{10}(\sigma)d\sigma \\
&\qquad \star\left\{\begin{array}{l}
+\int_0^t \mathcal{U}_1(\sigma)\alpha_2(\sigma)\cos(\omega_2\sigma)d\sigma - \int_0^t \mathcal{U}_2(\sigma)\alpha_2(\sigma)\sin(\omega_2\sigma)d\sigma \\
+\int_0^t \left[\int_0^\sigma \alpha_1(\eta)\sin(\omega_1\eta)d\eta\right]u_{20}(\sigma)d\sigma \\
-\int_0^t \left[\int_0^\sigma \alpha_2(\eta)\cos(\omega_2\eta)d\eta\right]u_{10}(\sigma)d\sigma
\end{array}\right. \\
&\qquad (I) \qquad + \int_0^t \left[\int_0^\sigma \alpha_1(\eta)\sin(\omega_1\eta)d\eta\right]\alpha_2(\sigma\cos(\omega_2\sigma)d\sigma \\
&\qquad (II) \qquad - \int_0^t \left[\int_0^\sigma \alpha_2(\eta)\cos(\omega_2\eta)d\eta\right]\alpha_1(\sigma)\sin(\omega_1\sigma)d\sigma
\end{aligned}$$

We claim the terms $*$ are small for large ω (based on the Riemann-Lebesgue lemma). Term (I) can be expanded to get

$$\begin{aligned}
&(I)\int_0^t \left[\int_0^\sigma \alpha_1(\eta)\sin(\omega_1\eta)d\eta\right]\alpha_2(\sigma\cos(\omega_2\sigma)d\sigma \\
&= \int_0^t \left(\frac{1}{\omega_1}\alpha_1(\eta)\cos(\omega_1\eta)\Big|_0^\sigma + \frac{1}{\omega_1}\int_0^\sigma \dot{\alpha}_1(\eta)\cos(\omega_1\eta)d\eta\right)\cdot\alpha_2(\sigma)\cos(\omega_2\sigma)d\sigma \\
&= \int_0^t \frac{1}{\omega_1}\left(\alpha_1(\sigma)\cos(\omega_1\sigma) - \alpha_1(0)\right)\left(\alpha_2(\sigma)\cos(\omega_2\sigma)d\sigma\right) \\
&\qquad + \frac{1}{\omega_1}\int_0^t\int_0^\sigma \dot{\alpha}_1(\eta)\cos(\omega_1\eta)d\eta\alpha_2(\sigma)\cos(\omega_2\sigma)d\sigma \\
&= \int_0^t \frac{1}{\omega_1}\alpha_1(\sigma)\alpha_2(\sigma)\cos(\omega_1\sigma)\cos(\omega_2\sigma)d\sigma - \int_0^t \frac{1}{\omega_1}\alpha_1(0)\alpha_2(\sigma)\cos(\omega_2\sigma)d\sigma + \frac{1}{\omega_1}\int \ldots
\end{aligned}$$

Now consider the point of view where the ratio $\frac{\alpha_1\alpha_2}{\omega_1}$ is important. What does the second term produce?

$$\int_0^t \frac{1}{\omega_2}\alpha_1(\sigma)\alpha_2(\sigma)\sin(\omega_1\sigma)\sin(\omega_2\sigma)d\sigma + \ldots$$

Combining this term with the matching term above and choosing $\omega_1 = \omega_2$ we have

$$\begin{aligned} x_3(t) &= \int_0^t \frac{1}{\omega_1}\alpha_1\alpha_2(\cos^2(\omega_1\sigma) + \sin^2(\omega_1\sigma))d\sigma + \ldots \\ &= \int_0^t \frac{1}{\omega_1}\alpha_1(\sigma)\alpha_2(\sigma)d\sigma + \int_0^t \mathcal{U}_1(\sigma)u_{20}(\sigma) - \mathcal{U}_2(\sigma)u_{10}(\sigma)d\sigma + stuff \end{aligned}$$

Now given x_{1d}, x_{2d} and x_{3d} how do we choose $u_1(t)$ and $u_2(t)$? Use the following

$$\begin{aligned} u_{10}(t) &= \frac{d}{dt}x_{1d}(t) \\ u_{20}(t) &= \frac{d}{dt}x_{2d}(t) \end{aligned}$$

We then have

$$\begin{aligned} \frac{1}{\omega_1}\int_0^t \alpha_1(\sigma)\alpha_2(\sigma)d\sigma &= x_{3d}(t) - \int_0^t \mathcal{U}_1(\sigma)u_{20}(\sigma) - \mathcal{U}_2(\sigma)u_{10}(\sigma)d\sigma \\ \frac{\alpha_1\alpha_2}{\omega_1} &= \frac{d}{dt}(\star) \end{aligned}$$

So we then have a number of choices, one of which is

$$\begin{aligned} \alpha_1 &= \sqrt{\omega_1}\sqrt{|\star|} \\ \alpha_2 &= \sqrt{\omega_1}\sqrt{|\star|}sgn(\star) \end{aligned}$$

As an example, consider the system where we want $x_{1d} = 0$, $x_{2d} = 0$ and $x_{3d} = t$. The equations of motion for the system (with the above controls) are

$$\begin{aligned} \dot{x}_1 &= u_{10}(t) + \alpha_1(t)\sin(\omega t) = 0 + \alpha_1(t)\sin(\omega t) \\ \dot{x}_2 &= u_{20}(t) + \alpha_2(t)\cos(\omega t) = 0 + \alpha_2(t)\cos(\omega t) \\ \dot{x}_3 &= x_1u_2 - x_2u_1 \end{aligned}$$

Choosing $\alpha_1 = \alpha_2 = \sqrt{\omega}$ we have

$$\begin{aligned} \dot{x}_1 &= \sqrt{\omega}\sin(\omega t) \\ \dot{x}_2 &= \sqrt{\omega}\cos(\omega t) \\ \dot{x}_3 &= 1 \end{aligned}$$

This choice of controls is referred to as an *approximate inverse* as it approximately inverts the system dynamics. For additional states of the form

$$\dot{x}_{ij} = x_iu_j - x_ju_i$$

simply choose different frequencies for each state that is to be controlled and sum the oscillatory components.

For higher order polynomial systems of the form

$$\begin{aligned}\dot{x}_i &= u_i \\ \dot{z}_j &= x_{k_1}\cdots x_{k_p}u_l\end{aligned}$$

note that the controls

$$\begin{aligned}u_l &= \alpha_l\cos(p\omega t) \\ u_{k_i} &= \alpha_{k_i}\sin(\omega t)\end{aligned}$$

will result in

$$\dot{z}_j = \alpha_l\alpha_{k_1}\cdots\alpha_{k_p}\underbrace{\cos^p(\omega t)\cos(p\omega t)}_{= const + trig}$$

As a general comment, these are not the only inputs that will generate motion for such systems, they are simply a choice that works. Only in the case of systems controllable with first level Lie brackets will the use of this type of sinusoid be optimal in any sense.

In order to use these results for systems that are not directly expressed in the form above, we can use the following result

Theorem 0.2 (Brockett) *Given $\dot{x} = B(x)u$ with $dim u = m$ and $dim x = m(m+1)/2$ and given that the vector fields and their first level Lie brackets span $\mathbb{R}^{m(m+1)/2}$ we can choose coordinates $(x^1, \ldots, x^m, y^{1,2}, \ldots, y^{m-1,m}$ in a neighborhood of a given point, say $x = 0$, so that the equations take the form*

$$\begin{aligned}\dot{x}^i &= u^i + r^i,\ i = 1, \ldots, m \\ \dot{y}^{ij} &= x^i u^j - x^j u^i + r^{ij},\ i, j = 1, \ldots, m,\ i < j\end{aligned}$$

where the r^i and r^{ij} have vanishing first partials with respect to x and y and in addition r^{ij} has vanishing second partials with respect to x^i and x^j.

Thus, we can locally transform nonlinear systems, controllable using first order Lie brackets, to the structure of a nonholonomic integrator in much the same way that we linearize nonlinear systems. The nonholonomic integrator structure can be thought of as a basis for nonlinear systems.