AA599: Geometric Methods for Nonlinear Control Systems
Lecture Notes
Observability

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1 Observability

READING: ISIDORI 2.3, NIJMEIJER AND VAN DER SCHAFT ??

The question to be answered in this section is the following. Given

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i,\ y = h(x) \tag{1}$$

can we find x_0 from a knowledge of $y(t)$ on the interval $t \in [0,1]$?

Consider first the linear time invariant system with no control ($u = 0$) where we assume x is C^∞:

$$\dot{x} = Ax,\ y = Cx$$

Differentiating the output gives

$$\left.\begin{array}{rcl} y & = & Cx \\ y^{(1)} & = & C\dot{x} = CAx \\ y^{(2)} & = & CA\dot{x} = CA^2x \\ & \vdots & \\ Y^{(n-1)} & = & CA^{(n-1)}x \end{array}\right\} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{(n-1)} \end{bmatrix} x = Nx.$$

In order to solve for $x(0)$, we must invert N and therefore require the observability matrix to be full rank. (Note that we know $y(t)$ and its derivatives)

Now, for linear systems if we allow nonzero control

$$\begin{array}{rcl} y & = & Cx \\ y^{(1)} & = & CAx + CBu \\ y^{(2)} & = & CA^2x + CBu + CABu \\ & \vdots & \end{array}$$

But if N is full rank, then u has no effect on observability. This result will not be true for nonlinear systems.

We will need to work with a Lie algebra formed of operators rather than vector fields (recall that differentiation is an operation). Specifically we are looking for:

> Given that $u = 0$, when can we find x_0 from $y(\cdot)$ on $[0, t]$, assuming that f_0 and h are real analytic?

To find x_0, take derivatives of the output as in the case of the linear system:

$$\begin{array}{rcllll} y(t) & = & h(x(t)) & \in \mathcal{Y} & (y) \\ y^{(1)}(t) & = & \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f_0(x) & \in T\mathcal{Y} & (y, \dot{y}) \\ y^{(2)}(t) & = & \frac{\partial}{\partial x}(\frac{\partial h}{\partial x} f_0(x)) f_0(x) & \in T^2\mathcal{Y} & (y, \dot{y}, \ddot{y}) \\ & \vdots & & & \\ y^{(n-1)}(t) & = & \frac{\partial}{\partial x}(\frac{\partial}{\partial x}(\cdots)) f_0(x) & & \end{array} \tag{2}$$

As we can see, we will get successively higher derivatives with respect to x, but not with respect to time. Define the following

$$\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(k)} \end{bmatrix} = \mathcal{F}(x)$$

This function defines a mapping from the state space $\mathcal{U}$ to the observation space $\mathcal{Y}$ as shown in Fig. 1. In order for the mapping to be invertible at a given point x_0, the Jacobian of $\mathcal{F}$ must have the same rank as the dimension of the state space at x_0.

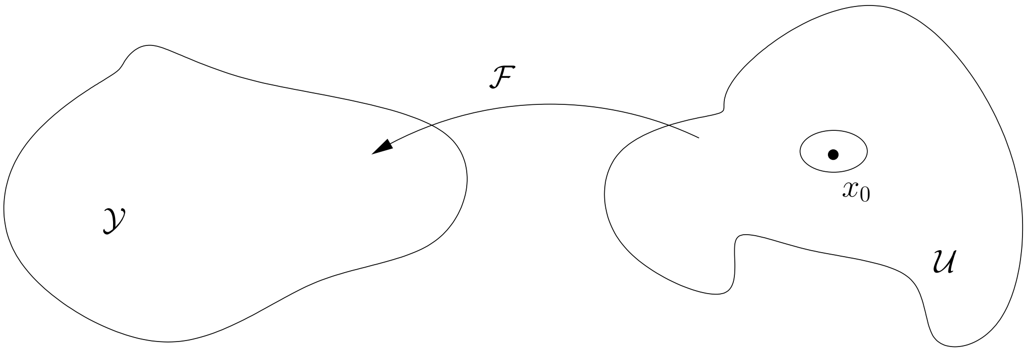

Figure 1: Mapping between the state and observation spaces.

Now, note that all of the terms in (2) come from the operator Lie algebra (meaning the terms are operators, and here they operate on the state space)

$$\left\{ h, \sum f_{0i} \frac{\partial}{\partial x_i} \right\}_{LA}$$

where f_{0i} are the components of the vector field f_0 and x_i are the coordinates of a local chart about x_0 on the state manifold $\mathcal{U}$. To see how this Lie algebra functions, consider $C^\infty(X)$ the set of infinitely differentiable functions on X. This collection has a Lie bracket. Let $\phi(x)$ be considered as an operator mapping $C^\infty(X) \to C^\infty(X)$ by $\psi \mapsto \phi\psi$. We will refer to this operation as multiplication by ϕ. We can represent ϕ as $\phi = \sum \alpha_{ij}(x)\left(\frac{\partial}{\partial x_i}\right)^i$. An example of the multiplication operation is then

$$\left(\sum \alpha_j(x)\frac{\partial}{\partial x}\right)\psi = \sum \alpha_j(x)\left(\frac{\partial \psi}{\partial x_j}\right) \in C^\infty(X)$$

The set of linear operators on $C^\infty(X)$ will form a Lie algebra under the Lie bracket operation $[\cdot,\cdot]$. where from the example we have

$$< \phi(x), \alpha\frac{\partial}{\partial x} >= \phi_x \alpha$$

where ϕ_x denotes the partial derivatives of ϕ.

In the case of the observation calculations, we will have $\phi(x) = h(x)$ and $\alpha(x) = f_0(x)$, and in particular recall that the vector field $f_0(x)$ may be treated as the operator $f_{0i}\frac{\partial}{\partial x_i}$. We then have

$$\begin{aligned} (hf_0 - f_0h)\psi &= h(x)\sum f_{0i}\frac{\partial \psi}{\partial x_i} - \sum f_i \frac{\partial}{\partial x_i}(h\psi) \\ &= -\psi(\sum f_{0i}\frac{\partial h}{\partial x_i}) \\ \Rightarrow (hf_0 - f_0h) &= \sum f_i \frac{\partial h}{\partial x_i} \in C^\infty(x) \end{aligned}$$

where this last result is a function on the manifold. We can then consider the *observation Lie algebra* $\{h, \sum f_{0i}\frac{\partial}{\partial x_i}\}_{LA} = \text{span}\{f_0, h, f_0h, f_0^2h, \ldots\}$. We want this algebra to span the state space at all points of the manifold which will allow us to invert the observation function and construct the state. Note that this will only be a local result as the inversion will depend on evaluation of the observation Lie algebra at a particular point of the manifold.

Now, these results give us information on how to determine controllability if the system inputs are identically zero. We need to determine what will happen if the controls are not zero, $u \neq 0$. We will use the following informal definitions:

Definition 1.1 *Given the nonlinear system (**??**, we will refer to the operator Lie algebra* $\{h, f_0\}_{LA}$ *as the* little observability algebra. *We will refer to* $\{h, f_0, f_i\}_{LA}$ *as the* big observability algebra.

Not that the big observability algebra contains $\{h, f_0 + \sum \alpha_i g_i\}_{LA}$ and also contains $\{h, f_0, [f_0, f_i]\}_{LA}$, etc. The criteria that we are looking for is that the functions in $\{h, f_0, f_i\}_{LA}$ can be organized into a Jacobian of rank equal to the dimension of the manifold at all points of the manifold (or at least in some submanifold). This result is referred to as the observability rank condition at p_0 and simply the observability rank condition if it holds at all points $p_0 \in \mathcal{U}$.

Now we will formalize these ideas. To begin we will make some definitions (see [**?**]).

Definition 1.2 *Two states* $x_1, x_2 \in \mathcal{U}$ *are said to be* indistinguishable *(denoted* $x_1 I x_2$*) for* Σ *if for every admissible input function* u *the output function* $t \mapsto y(t, 0, x_1, u), t \geq 0$ *of the system for initial state* $x(0) = x_1$ *and the output function* $t \mapsto y(t, 0, x_2, u), t \geq 0$ *of the system for initial state* $x(0) = x_2$ *are identical on their common domain of definition. The system is called observable if* $x_1 I x_2$ *implies* $x_1 = x_2$.

Definition 1.3 *The system* Σ *is called* locally observable *at* x_0 *if there exists a neighborhood* W *of* x_0 *such that for every neighborhood* $V \subset W$ *of* x_0 *the relation* $x_0 I^V x_1$ *implies that* $x_1 = x_0$. *If the system is locally observable at each* x_0 *then it is called* locally observable.

Definition 1.4 *The observation space* $\mathcal{O}$ *of* Σ *is the linear space (over* $\mathbb{R}$*) of functions on* $\mathcal{U}$ *containing* $h_1, \ldots, h_p$ *and all repeated Lie derivatives*

$$L_{X_1} L_{X_2} \cdots L_{X_k} h_j, j \in p, k = 1, 2, \ldots$$

with X_i *in the set* $\{f_0, f_1, \ldots, f_m\}$.

Proposition 1.5 $\mathcal{O}$ *is also the linear space of functions on* $\mathcal{U}$ *containing* $h_1, \ldots, h_p$ *and all repeated Lie derivatives*

$$L_{Z_1} \cdots L_{Z_k} h_j, j \in \{1, \ldots, p\}, k = 1, 2, \ldots$$

with $Z_i, i \in \{1, \ldots, k\}$ *of the form*

$$Z_i(x) = f_0(x) + \sum_{j=1}^{m} f_i(x) u_j^i$$

for some point $(u_1^i, \ldots, u_m^i) \in \mathcal{K}$.

The observation space $\mathcal{O}$ defines the *observability codistribution* denoted $d\mathcal{O}$

$$d\mathcal{O}(q) = \text{span}\{dH(q) | H \in \mathcal{O}\}, q \in \mathcal{U}$$

Theorem 1.6 *Consider the system* Σ *with dim*$\mathcal{U} = n$. *Assume that dimd*$\mathcal{O}(x_0) = n$, *then the system is locally observable at* x_0.

Proof: Since $\dim\mathcal{O}(x_0) = n$ there exists n functions $H_1, \ldots, H_n \in \mathcal{O}$ such that $dH_1, \ldots dH_n(x_0)$ are linearly independent. Define the map $\Phi : \mathcal{U} \to \mathbb{R}^n$ as

$$\Phi(x) = (H_1(x) \cdots H_n(x))^T$$

It follows that the Jacobian matrix of Φ at x_0 is non-singular and therefore there exists a neighborhood W of x_0 such that $\Phi : W \to \Phi(W)$ is a diffeomorphism. Now let $V \subset W$ be a neighborhood of x_0 and suppose that $x_0 I^V x_1$ for some $x_1 \in V$. Then for any $i \in \{1, \ldots, p\}$ and $k \geq 0$ and for small $t_1, \ldots, t_k$ we have

$$h_i(Z_k^{t_k} \circ Z_{k-1}^{t_{k-1}} \circ \cdots \circ Z_1^{t_1}(x_0)) = h_i(Z_k^{t_k} \circ \cdots \circ Z_1^{t_1}(x_1))$$

with Z_i, $i \in k$ of the form $(\cdot)$. Differentiating both sides with respect to $t_k, t_{k-1}, \cdots, t_1$ (in order) at repsectively $t_k = 0, t_{k-1} = 0, \ldots$ gives

$$L_{Z_1} L_{Z_2} \cdots L_{Z_k} h_i(x_0) = L_{Z_1} L_{Z_2} \cdots L_{Z_k} h_i(x_1)$$

for all Z_j, $j \in \{1, \ldots, k\}$.

It follows that $H(x_0) = H(x_1)$ for all $H \in \mathcal{O}$. In particular, $H_i(x_0) = H(x_1)$, $i \in \{1, \ldots, n\}$. By injectivity of Φ we have $x_0 = x_1$. ■

Example 1.7 *Consider the system*

$$\begin{aligned} \dot{x} &= u, \ x \in \mathbb{R} \\ y_1 &= \sin(x) \\ y_2 &= \cos(x) \end{aligned}$$

Note that the linearization of the system with both outputs will be observable at all points on the manifold. However, with just the observation y_1, the resulting linearized system is not observable at the points $x = 2\pi k + \frac{\pi}{2}$. However the nonlinear observability condition gives

$$dim d\mathcal{O} = dim \ span\{cos(x)dx, sin(x)dx\}$$

in either case, which has rank one for all $x \in \mathbb{R}$. So the nonlinear system is always locally observable. It is not globally observable since $x_0 = x_1$ when $x_0 - x_1$ is a multiple of 2π.

Example 1.8 *Consider the system*

$$\dot{x} = 0, \ y = x^3, \ x \in \mathbb{R}$$

This system is obviously observalbe. We have $\mathcal{O} = \{x^3\}$ and dim $d\mathcal{O}(0) = 0$.

Example 1.9 *Consider the system*

$$\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j, \ y_i = C_i x$$

The observability algebra is $\mathcal{O} = span\{c_i x, c_i A x, \ldots, c_i A^{n-1} x\}, i \in \{1, \ldots, p\}$. Recall that $d\mathcal{O} = span\{c_i, c_i A, \ldots, c_i A^{n-1}\}$ which is invertible if

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$