AA599: Geometric Methods for Nonlinear Control Systems
Notes: April 6, 2004

For a system of the form $\dot{x} = f(x)$ (where $\ddot{x} = \frac{\partial f}{\partial x}\dot{x} = \frac{\partial f}{\partial x} f(x)$), the Taylor expansion about any point is

$$\begin{aligned} x(t) &= x(a) + \dot{x}(a)(x-a) + \ddot{x}\frac{1}{2!}(x-a)^2 + h.o.t. \\ &= x(0) + f(0)t + \frac{\partial f}{\partial x} f(0)\frac{t^2}{2} + h.o.t. \end{aligned}$$

where we have chosen $a = 0$ for simplicity.
Now consider switching controls for the system $\dot{x} = f_1(x)u_1 + f_2(x)u_2$ of the form

$$\begin{aligned} &u_1 = 1,\ u_2 = 0,\ 0 \le t < \epsilon \\ &u_1 = 0,\ u_2 = 1,\ \epsilon \le t < 2\epsilon \\ &u_1 = -1,\ u_2 = 0,\ 2\epsilon \le t < 3\epsilon \\ &u_1 = 0,\ u_2 = -1,\ 3\epsilon \le t < 4\epsilon \end{aligned}$$

Using the Taylor expansion, the system response at time $t = \epsilon$ is

$$x(\epsilon) = x_0 + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0) + h.o.t$$

where we will be assuming that $\epsilon \ll 1$ and neglect all terms of order more than two in ϵ. At time $t = 2\epsilon$ we use the end point of $x(\epsilon)$ as the initial condition for another evolution over time ϵ to get

$$\begin{aligned} x(2\epsilon) &= x(\epsilon) + \epsilon f_2(x(\epsilon)) + \epsilon^2 \frac{1}{2}\frac{\partial f_2}{\partial x} f_2(x(\epsilon)) + h.o.t \\ &= x_0 + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0) \\ &\quad + \epsilon f_2\left(x_0 + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0)\right) \\ &\quad + \epsilon^2 \frac{1}{2}\frac{\partial f_2}{\partial x} f_2\left(x_0 + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0)\right) + h.o.t. \end{aligned}$$

Now note that we can use the Taylor expansion again to rewrite the term f_2:

$$f_2\left(x_0 + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0)\right) = f_2(x_0) + \frac{\partial f_2}{\partial x}\left[\epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0)\right]$$

which gives (neglecting terms of order ϵ^3 and higher)

$$x(2\epsilon) = x(0) + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0) + \epsilon f_2(x_0) + \epsilon^2 \frac{\partial f_2}{\partial x} f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_2}{\partial x} f_2(x_0)$$

Now reverse the first control (note that $\frac{\partial}{\partial x}(-f) = -\frac{\partial f}{\partial x}$), and repeat the above process to get

$$\begin{aligned} x(3\epsilon) &= x(2\epsilon) - \epsilon f_1(x(2\epsilon)) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x(2\epsilon)) + h.o.t \\ &= x(0) + \epsilon f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0) + \epsilon f_2(x_0) + \epsilon^2 \frac{\partial f_2}{\partial x} f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_2}{\partial x} f_2(x_0) \\ &\quad - \epsilon\left[f_1(x_0) + \frac{\partial f_1}{\partial x}\epsilon(f_1(x_0) + f_2(x_0))\right] + \epsilon^2 \frac{1}{2}\frac{\partial f_1}{\partial x} f_1(x_0) \\ &= x(0) + \epsilon f_2(x_0) + \epsilon^2 \frac{\partial f_2}{\partial x} f_1(x_0) + \epsilon^2 \frac{1}{2}\frac{\partial f_2}{\partial x} f_2(x_0) - \epsilon^2 \frac{\partial f_1}{\partial x} f_2(x_0) \end{aligned}$$

After the final step we have

$$
\begin{aligned}
x(4\epsilon) &= x(3\epsilon) - \epsilon f_2(x(3\epsilon)) + \epsilon^2 \frac{1}{2} \frac{\partial f_2}{\partial x} f_2(x(3\epsilon)) + h.o.t \\
&= x(0) + \epsilon f_2(x_0) + \epsilon^2 \frac{\partial f_2}{\partial x} f_1(x_0) + \epsilon^2 \frac{1}{2} \frac{\partial f_2}{\partial x} f_2(x_0) - \epsilon^2 \frac{\partial f_1}{\partial x} f_2(x_0) \\
&\quad - \epsilon \left[f_2(x_0) + \frac{\partial f_2}{\partial x} \epsilon f_2(x_0) \right] + \epsilon^2 \frac{1}{2} \frac{\partial f_2}{\partial x} f_2(x_0) \\
&= x_0 + \epsilon^2 \left[\frac{\partial f_2}{\partial x} f_1(x_0) - \frac{\partial f_1}{\partial x} f_2(x_0) \right] \\
&= x_0 + \epsilon^2 \left[f_1, f_2 \right] (x_0)
\end{aligned}
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