AA599: Geometric Methods for Nonlinear Control Systems
Homework 7
Solutions

1. Khalil 3.3

(a) Choose the domain to be $D = \{x_1, x_2 | x_1 < 1\}$ and $V(x) = x^T \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} x$. Then $\dot{V} = -2kx_1^2 - 2k(1-x_1)x_2^2$ which is negative on D. This system has only one equilibrium and must therefore be globally asymptotically stable.

(b) Choose the domain to be $D = \{x_1, x_2 | x_1^2 + x_2^1 < 1\}$ and $V(x) = x^T \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} x$ with $k > 0$. Then $\dot{V} = 2k(-1 + x_1^2 + x_2^2)(x_1^2 + x_2^2) < 0$. The equilibria for this system are given by $x_1 = x_2 = 0$ and the set $\{x_1, x_2 | x_1^2 + x_2^2 = 1\}$, so the system is not globally asymptotically stable.

(c) Choose the domain to be $D = \{x_1, x_2 | x_1^2 + x_2^2 < \frac{1}{\sqrt{2}}\}$ and $V(x) = x^T \begin{bmatrix} k & \frac{1}{2}k \\ \frac{1}{2}k & k \end{bmatrix} x$. Then $\dot{V} = -kx_1^2(1 - 2x_1x_2) - x_2^2(2 - x_1^2)$ which is strictly negative on D. The equilibria for this system are given by $x_1 = x_2 = 0, x_1 = x_2 = 1, x_1 = x_2 = -1$. The equilibrium $x_1 = x_2 = 1$ is also asymptotically stable and therefore the system is not globally asymptotically stable.

(d) Choose the domain to be $D = \mathbb{R}^2$ and $V(x) = x^T \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} x$. Then $\dot{V} = -2kx_1^2 - 2kx_2^4 < 0$. The equilibria for this system are given by $x_1 = x_2 = 0$, $x_1 = i, x_2 = -i$ and $x_1 = -i, x_2 = i$. The Lyapunov function is radially unbounded, so this system is globally asymptotically stable.

2. Khalil 3.18

(a) The state equations in the new coordinates are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1)x_2 - g(x_1) \end{aligned}$$

The equilibria of this system are given by $x_2 = 0$ and $-h(x_1)x_2 - g(x_1) = 0$ which can be met for any $h(x_1)$ and $g(x_1) = 0$. To ensure a single equilibrium with $g(0) = 0$, we must have $g(x_1)$ monotonic.

(b)

$$\begin{aligned} \dot{V} &= \frac{\partial}{\partial x_1} \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2 \\ &= (g(x_1) - g(0))\dot{x}_1 + x_2\dot{x}_2 \\ &= g(x_1)x_2 - h(x_1)x_2^2 - g(x)x_2 \\ &= -h(x_1)x_2^2 \end{aligned}$$

so we must have $h(x_1) > 0$.

(c)

$$\begin{aligned}
\dot{V} &= \left(x_2 + \int_0^{x_1} h(y)dy\right) [h(x_1) - h(0)\ 1] \begin{bmatrix} x_2 \\ -h(x_1)x_2 - g(x_1) \end{bmatrix} + (g(x_1) - g(0))x_2 \\
&= \left(x_2 + \int_0^{x_1} h(y)dy\right) (x_2(h(x_1) - h(0)) - h(x_1)x_2 - g(x_1)) + g(x_1)x_2 \\
&= -x_2^2 h(0) - (x_2 h(0) + g(x_1)) \int_0^{x_1} h(y)dy
\end{aligned}$$

For this function to be negative definite, we must have $h(0) \geq 0$ and $(x_2h(0)+g(x_1)) \int_0^{x_1} h(y)dy > 0$. These conditions can be met nicely if we have $h(0) = 0$, $h(x) \geq 0$ and $g(x) > 0$.

3. Khalil 3.24

We are given the Lyapunov equation

$$PA + A^T P = -C^T C$$

where the pair A, C corresponds to an observable system

$$\dot{x} = Ax,\ y = Cx.$$

We are to show that A is Hurwitz (real part of all eigenvalues in the left half-plane), if and only if there exists $P = P^T > 0$ that satisfies the equation. Also, we are to show that if A is Hurwitz, the Lyapunov equation will have a unique solution.

To begin, we will show the "if" (sufficiency) direction, meaning that we assume the solution to the Lyapunov function exists. We propose the Lyapunov function $V(x) = y^T \tilde{P} y$ which has derivative

$$\begin{aligned}
\dot{V}(x) &= \dot{y}^T \tilde{P} y + y^T \tilde{P} \dot{y} \\
&= \dot{x}^T C^T \tilde{P} C x + x^T C^T \tilde{P} C \dot{x} \\
&= x^T A^T C^T \tilde{P} C x + x^T C^T \tilde{P} C A x \\
&= x^T \left(A^T C^T \tilde{P} C + C^T \tilde{P} C A\right) x
\end{aligned}$$

Definining $P = C^T \tilde{P} C$ we have

$$\begin{aligned}
\dot{V}(x) &= x^T \left(A^T P + PA\right) x \\
&= -x^T C^T C x
\end{aligned}$$

If $C^T C > 0$, then by Lyapunov, the system must be stable (A Hurwitz). If $C^T C$ is not positive definite (such as if $C = [1, 0, 0, \ldots]$), then we must apply LaSalle's theorem. This is a fairly straightforward step, as the system is linear, so if it is stable and observable, the only way to have $Cx = 0$ is if the entire trajectory is identically zero meaning we are at an equilibrium point. Therefore we have shown sufficiency.

For the only if direction (necessity), consider the matrix P defined by

$$P = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

The integral exists because the integrand is a sum of terms of the form $t^{k-1}e^{\lambda_i t}$ with $Re\lambda_i < 0$. To show that P is positive definite, we use proof by contradiction. Assume that P is not positive definite. then there is a vector $x \neq 0$ such that $x^T P x = 0$. However,

$$\begin{aligned} x^T P x = 0 \quad &\Rightarrow \quad \int_0^\infty x^T e^{A^T t} C^T C e^{At} x dt = 0 \\ &\Rightarrow \quad C e^{At} x \equiv 0, \forall t \geq 0 \Rightarrow x = 0 \end{aligned}$$

since e^{At} is nonsingular for all t and the system is observable. So P is positive definite. We can now write

$$\begin{aligned} PA + A^T P \quad &= \quad \int_0^\infty e^{A^T t} C^T C e^{At} A dt + \int_0^\infty A^T e^{A^T t} C^T C e^{At} dt \\ &= \quad \int_0^\infty \frac{d}{dt} e^{A^T t} C^T C e^{At} dt = e^{A^T t} C^T C e^{At} \Big|_0^\infty = -C^T C \end{aligned}$$

Therefore, P is a solution of the Lyapunov equation. For brevity I will not show that it is the only solution, but to do so simply assume another exists, $\tilde{P}$ and show that this gives a contradiction.

4. Khalil 3.25

(a)

$$\begin{aligned} \dot{V} \quad &= \quad \dot{x}^T P x + x^T P \dot{x} \\ &= \quad x^T (A - BR^{-1}B^T P)^T P x + x^T P (A - BR^{-1}B^T P) x \\ &= \quad x^T (A^T P - PBR^{-1}B^T P + PA - PBR^{-1}B^T P) x \\ &= \quad -x^T (Q + PBR^{-1}B^T P) x \\ &< \quad 0 \text{ (because pos def + pos semi = pos def)} \end{aligned}$$

(b)

$$\dot{V} \quad = \quad -x^T C^T C x \leq 0$$

We must characterize the set where equality holds, then apply LaSalle's theorem. Obviously equality holds iff $Cx = 0$. Using the hint with $t = 0$, for an observable system this can only happen if $x = 0$. Therefore the system is stable.

5. Consider the control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i$$

with $x(t)$ taking on values in a differentiable manifold X. Assume that the system is controllable. Show that the control u that drives the system from the state x_0 to the state x_1 in one unit of time and minimizes

$$\eta = \int_0^t \sum_{i=1}^{m} u_i^2 dt$$

is such that $\|u(t)\|$ is constant on the interval $[0, 1]$.

Rather than a Lagrangian formulation, we will use a Hamiltonian approach (known as the Maximum Principle). For this approach, we will solve the extremal problem formulated as a maximum rather than a minimum:

$$\eta = \int_0^1 -\sum_{i=1}^m u_i^2 dt$$

The appropriate Hamiltonian for this problem is then

$$\begin{aligned} H &= p^T \sum_{i=1}^m g_i(x) u_i - p_{m+1} \sum_{i=1}^m u_i^2 \\ &= \sum_{i=1}^m \left(p^T g_i(x) u_i - p_{m+1} u_i^2 \right) \\ &= \sum_{i=1}^m \left(p^T g_i(x) - p_{m+1} u_i \right) u_i \end{aligned}$$

The optimal control, $u^\star$, for this problem must satisfy $\frac{\partial H}{\partial u^*} = 0$:

$$\frac{\partial H}{\partial u} = \sum_{i=1}^m \left(p^T g_i(x) - 2 p_{m+1} u_i \right) = 0$$

and

$$u_i = \frac{p^T g_i(x)}{2 p_{m+1}}$$

Now if $\|u\|$ is constant, its time derivative will be zero:

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^m u_i^2 &= \frac{d}{dt} \sum_{i=1}^m \frac{\left(p^T g_i(x) \right)^2}{4 p_{m+1}^2} \\ &= \frac{1}{2 p_{m+1}^2} \sum_{i=1}^m \left(p^T g_i(x) \right) \left(\dot{p}^T g_i(x) + p^T \dot{g}_i(x) \right) \end{aligned}$$

Recall that the costates p evolve according to

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial x} \\ &= -p^T \sum_{i=1}^m \frac{\partial g_i}{\partial x} u_i \end{aligned}$$

We then have

$$\begin{aligned} \frac{d}{dt} u_i^2 &= \frac{1}{2 p_{m+1}^2} \sum_{i=1}^m \left(p^T g_i(x) \right) \left(- \left(p^T \sum_{j=1}^m \frac{\partial g_j}{\partial x} u_j \right)^T g_i(x) + p^T \frac{\partial g_i}{\partial x} \sum_{j=1}^m g_j(x) u_j \right) \\ &= 0 \end{aligned}$$