AA599: Geometric Methods for Nonlinear Control Systems
Homework # 5
Solutions

1. **Consider a frictionless, rigid two-link robot manipulator (or double pendulum) with control torques u_1 and u_2 applied at the joints (see Fig. ??). The equations of motion for this system are given by**

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + k(\theta) = u$$

where

$$\begin{aligned}
M(\theta) &= \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos(\theta_2) & m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_2) \\ m_2 l_2^2 + m_2 l_1 l_2 \cos(\theta_2) & m_2 l_2^2 \end{bmatrix} \\
C(\theta,\dot{\theta}) &= \begin{bmatrix} -m_2 l_1 l_2 \ \sin(\theta_2)\ \dot{\theta}_2\ (2\dot{\theta}_1 + \dot{\theta}_2) \\ m_2 l_1 l_2 \ \sin(\theta_2)\ \dot{\theta}_1^2 \end{bmatrix} \\
k(\theta) &= -\begin{bmatrix} m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) + m_2 g l_2 \sin(\theta_1 + \theta_2) \\ m_2 g l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}
\end{aligned}$$

The determinant of M is positive for all θ and therefore the equations can be rewritten as

$$\ddot{\theta} = -M(\theta)^{-1}C(\theta,\dot{\theta}) - M(\theta)^{-1}k(\theta) + M(\theta)^{-1}u.$$

Let the output for this system be the angle of the first joint:

$$y = \theta_1.$$

Linearize this system about $\theta_1 = \theta_2 = \dot{\theta}_1 = \dot{\theta}_2 = 0$ and $u_1 = u_2 = 0$. Show that the linearized system is observable for $g \neq 0$, while it is not observable for $g = 0$. On the other hand, show that $\dim d\mathcal{O} = 4$ even in the case $g = 0$.

The linearized system is given by

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-gm_2+g(m_1+m_2)}{l_1 m_1} & -\frac{gm_2}{l_1 m_1} & 0 & 0 \\ -\frac{g}{l_1} & \frac{g(l_2 m_2 + l_1(m_1+m_2))}{l_1 l_2 m_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{l_1^2 m_1} & \frac{l_1+l_2}{l_1^2 l_2 m_1} \\ -\frac{l_1+l_2}{l_1^2 l_2 m_1} & \frac{2l_1 l_2 m_2 + l_2^2 m_2 + l_1^2(m_1+m_2)}{l_1^2 l_2^2 m_1 m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}$$

For the linear system, the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-gm_2+g(m_1+m_2)}{l_1 m_1} & -\frac{gm_2}{l_1 m_1} & 0 & 0 \\ 0 & 0 & \frac{-gm_2+g(m_1+m_2)}{l_1 m_1} & -\frac{gm_2}{l_1 m_1} \end{bmatrix}$$

which is obviously full rank as long as $g \neq 0$.

For the nonlinear system, we have the observability algebra $\mathcal{O} = \{h, L_f h, L_f^2 h, L_f^3 h\}$. Computing $d\mathcal{O}$ we have full rank at all points except when $g = 0$ as with the linearized system. In order to show observability for all points, we will use the following result (from Nijmeijer and van der Schaaft)

Proposition 0.1 *Let the nonlinear system*

$$\begin{aligned} \dot{x} &= f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \;\; x \in \mathcal{U} \\ y &= h(x) \end{aligned}$$

be a locally accessible and analytic system Furthermore, suppose that $\mathcal{U}$ is connected. Then the codistribution $d\mathcal{O}$ is constant-dimensional. In particular, the system is locally observable if and only if it satisfies the observability rank condition.

So we simply need to show that the system is connected and accesible to konw that the observability algebra must be constant rank. As we know that the rank is four at most points, it would then need to be four everywhere. To see that the system is accessible, we can use the set of vector fields f_1, f_2, $[f_0, f_1]$ and $[f_0, f_2]$. The dimension of the span of these vector fields can be shown to be

$$dim \begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & M^{-1}(\theta) \\ M^{-1}(\theta) & \begin{matrix} \star & \star \\ \star & \star \end{matrix} \end{bmatrix}$$

which is full rank as the mass matrix is necessarily invertible. Therefore the system is observable.

2. **Is the system from problem 3 of homework 3 observable if U is zero and one observes the trace of A?**

   With $U = 0$ and $h(x) = tr(A)$, the observability algebra is given by

$$\mathcal{O} = \left\{ h(x), L_f h(x), L_f^2 h(x), \ldots \right\}$$

   where

$$\begin{aligned} h &= tr(A) = tr(\Omega^0 A) \\ L_f h &= tr(\dot{A}) = tr(\Omega A) \\ L_f^2 h &= tr(\dot{\Omega} A) + tr(\Omega \dot{A}) = tr(I \left(\left(\Omega I^{-1} \right)^2 - \left(I^{-1} \Omega \right)^2 \right) I A) + tr(\Omega^2 A) \\ L_f^3 h &= (\ldots) + tr(\Omega^3 A) \\ L_f^4 h &= (\ldots) + tr(\Omega^4 A) \\ L_f^5 h &= (\ldots) + tr(\Omega^5 A) \end{aligned}$$

To show that the system is observable we must show that

$$\dim \text{span}\{dh, dL_f h, dL_f^2 h, \ldots\} = 6$$

This test effectively reduces to checking that

$$\dim \text{span}\{d(tr(\Omega^0 A)), d(tr(\Omega^1 A)), d(tr(\Omega^2 A)), \ldots\} = 6$$

Parameterizing A such as with Euler-angles, and performing the computations will give the desired result. A key point here is the following

$$\begin{aligned} tr(\Omega^3 A) &= tr(\Omega A)(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ tr(\Omega^4 A) &= tr(\Omega^2 A)(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ tr(\Omega^5 A) &= tr(\Omega A)(\omega_1^2 + \omega_2^2 + \omega_3^2)^2 \end{aligned}$$

Using these terms in conjunction with the Lie derivatives will give $d\mathcal{O}$ having full rank.