AA599: Geometric Methods for Nonlinear Control Systems
Homework # 4

1. **Consider a system defined on a $n(n+1)/2$-dimensional vector space, realized as pairs (x, Z) where x is an n-vector and $Z = -Z^T$ is a skew-symmetric matrix. The equations of motion are**

$$\begin{aligned} \dot{x} &= u \\ \dot{Z} &= xu^T - ux^T \end{aligned}$$

Show that this system is controllable.

One way to approach this problem is to rewrite the equations as

$$\begin{aligned} \dot{x}_i &= u_i, i \in \{1, \dots, m\} \\ \dot{z}_{ij} &= x_i u_j - x_j u_i,\ i < j, i, j \in \{1, \dots, m\} \end{aligned}$$

Rewriting the system in control affine form, we have

$$\frac{d}{dt}\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \sum_{i=1}^{m}\left(e_i + \sum_{j=1}^{m} e_{ij}x_i - e_{ji}x_j\right)u_i = \sum_{i=1}^{m} f_i(x)u_i$$

where e_i is the vector of zeros with a 1 in the i-th location and e_{ij} is the vector of zeros with a 1 in the ij-th location. Taking Lie brackets of the control vector fields, we have

$$[f_i, f_j] = 2e_{ij}$$

The collection of the f_i with the Lie brackets clearly spans the tangent space, and thus the system is controllable.