CDS 221: Control of Nonlinear Systems
Homework #3
Solutions

1. **Describe the reachable set as a functin of the time t for the equation**

$$\dot{p} = -2p + 1 - p^2 u$$

under the hypothesis that $p(0) = 1$ and $u \geq 0$. Repeat this for the matrix equation

$$\dot{P} = AP + PA^T + I - PUP$$

under the assumption that $P(0) = I$ and $U = U^T \geq 0$.

First for the scalar system, note that under just the drift we have

$$\dot{p} = -2p + 1$$

which has the solution

$$p(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}p(0)$$

and therefore from the given initial condition we can reach any point in $[\frac{1}{2}, 1]$ where the system will begin at 1 and flow to $\frac{1}{2}$. If we now take into consideration the effect of the control, we note that p^2 is always positive and with $u \geq 0$, we can only cause the system to decay faster and never increase past a value of 1. Because $p(t)$ can never be negative, we can now reach any point in the set $[0, 1]$.

Now for the matrix equation, note that the unforced equation is given by

$$\dot{P} = AP + PA^T + I$$

and assume that P_0 satisfies $AP_0 + P_0A^T + I = 0$. Then the solution to the unforced equation is given by

$$P_u(t) = e^{At}\left(P_u(0) - P_0\right)e^{A^Tt} + P_0$$

where we are assuming that the eigenvalues of A are strictly negative (so that this solution converges). The flow of these equations is such that $I \geq P_u(t) \geq X > 0$ for some some matrix X. As with the scalar system, when we apply the control it enters in strictly positively. As we can make it as large as we desire, the set of reachable matrices is $I \geq P(t) \geq 0$. In the case that A has one or more unstable eigenvalues, then the equations will tend to grow to any value. If unbounded controls are allowed, then the value of the state can be kept within any level, and the reachable values are $P \geq 0$.

2. **The equations of motion for a rigid body with applied torque can be expressed in terms of the angular velocities in a body-fixed coordinate frame. If the frame correspdonds to principal axes then the equations are**

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + u_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_1\omega_3 + u_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 + u_3 \end{aligned}$$

Introduce the matrix

$$\Omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

Express these equations as a quadratic matrix equation in Ω making use of the notation

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix},\ U = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix}$$

To begin, note that

$$I^{-1}\Omega = \begin{bmatrix} 0 & -\frac{\omega_3}{I_1} & \frac{\omega_2}{I_1} \\ \frac{\omega_3}{I_2} & 0 & -\frac{\omega_1}{I_2} \\ -\frac{\omega_2}{I_3} & \frac{\omega_1}{I_3} & 0 \end{bmatrix},\ \Omega I^{-1} = \begin{bmatrix} 0 & -\frac{\omega_3}{I_2} & \frac{\omega_2}{I_3} \\ \frac{\omega_3}{I_1} & 0 & -\frac{\omega_1}{I_3} \\ -\frac{\omega_2}{I_1} & \frac{\omega_1}{I_2} & 0 \end{bmatrix}$$

Then the Euler equations are equivalent to

$$\begin{aligned} \dot{\Omega} &= I\left(\left(\Omega I^{-1}\right)^2 - \left(I^{-1}\Omega\right)^2\right) I + U \\ &= f_0(\Omega) + g(U) \end{aligned}$$

3. **The above equations can be augmented by the differential equation of the direction cosines associated with the principal axes A to get**

$$\begin{aligned} \dot{\Omega} &= f_0(\Omega) + g(U) \\ \dot{A} &= \Omega A \end{aligned}$$

Are these equations controllable? If u_1 is constrained to be zero does the answer change?

Define the basis functions

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ J_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that the functions J_1A, J_2A and J_3A determine rotations of A about each of its three axes and therefore define basis functions for A. Expressing the drift and control is a bit more clear using the original description of the orientations with ω_1, ω_2, ω_3 and the corresponding basis functions $\frac{\partial}{\partial\omega_1}$, $\frac{\partial}{\partial\omega_2}$, and $\frac{\partial}{\partial\omega_3}$. We then have

$$\begin{aligned} f_0(\Omega, A) &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3\frac{\partial}{\partial\omega_1} + \frac{I_3 - I_1}{I_2}\omega_1\omega_3\frac{\partial}{\partial\omega_2} + \frac{I_1 - I_2}{I_3}\omega_1\omega_2\frac{\partial}{\partial\omega_3} + \omega_1 J_1 A + \omega_2 J_2 A + \omega_3 J_3 A \\ f_1(\Omega, A) &= \frac{\partial}{\partial\omega_1} \\ f_2(\Omega, A) &= \frac{\partial}{\partial\omega_2} \\ f_3(\Omega, A) &= \frac{\partial}{\partial\omega_3} \end{aligned}$$

We will show controllability for $u_1 = 0$ and therefore for $u_1 \neq 0$. From the following

$$\begin{aligned}
[f_0, f_2] &= -\frac{I_2 - I_3}{I_1}\omega_3 \frac{\partial}{\partial \omega_1} - \frac{I_1 - I_2}{I_3}\omega_1 \frac{\partial}{\partial \omega_3} + J_2 A \\
[f_3, [f_0, f_2]] &= -\frac{I_2 - I_3}{I_1} \frac{\partial}{\partial \omega_1} \\
[f_0, f_3] &= -\frac{I_2 - I_3}{I_1}\omega_2 \frac{\partial}{\partial \omega_1} - \frac{I_3 - I_1}{I_2}\omega_1 \frac{\partial}{\partial \omega_2} - J_3 A \\
[[f_0, f_2], [f_0, f_3]] &= \left(\frac{I_2 - I_3}{I_1}\right)\left(\frac{I_3 - I_1}{I_2}\right)\omega_3 \frac{\partial}{\partial \omega_2} - \left(\frac{I_2 - I_3}{I_1}\right)\left(\frac{I_1 - I_2}{I_3}\right)\omega_2 \frac{\partial}{\partial \omega_3} - J_1 A
\end{aligned}$$

we see that the directions $\{\frac{\partial}{\partial \omega_1}, \frac{\partial}{\partial \omega_2}, \frac{\partial}{\partial \omega_3}, J_1 A, J_2 A, J_3 A\}$ are independently generated as long as $I_2 \neq I_3$. Thus the system is controllable.