AA599: Geometric Methods for Nonlinear Control Systems
Homework #2
Solutions

1. **Find the distribution generated by the vector fields in $\mathbb{R}^3$**

$$F_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ F_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

Find a manifold in $\mathbb{R}^3$ that includes the point $(1,0,0) = (x_0, y_0, z_0)$ and has a tangent space spanned by these vector fields. (Please be careful about singular points if there are any.)

$$\begin{aligned}\Delta(x) &= \text{span}\{F_1, F_2\}\\ &= \text{span}\left\{\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}\right\}\end{aligned}$$

The dimension of this distribution is 0 at the origin, 1 for $x = 0$, $xz = 0$ or $yx = 0$ and is otherwise 2.

To find the manifold, note that the Lie bracket of these two vector fields is

$$[F_1, F_2] = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}$$

At the point $(1,0,0)$, the three vector fields are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the manifold is taken to be S^2, the vector fields F_1, F_2 and $[F_1, F_2]$ will have dimension 2 everywhere, and the requirements are met.

2. **Find the matrix Lie algebra generated by A and B**

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because the elements of the Lie algebra are in $\mathbb{R}^{n\times n}$ the total dimension of the Lie algebra will be $n^2 - 9$. So we need nine independent 3×3 matrices. Using Matlab, these matrices can be found to be

$$\begin{aligned}\{A, B\}_{LA} = \{&A,\ B,\ [A,B],\ [A,[A,B]],\ [B,[A,B]],\\ &[A,[A,[A,B]]],\ [A,[B,[A,B]]],\ [A,[A,[A,[A,B]]]],\ [A,[A,[B,[A,B]]]]\}\end{aligned}$$

To check the independence of the matrices, one can treat them as vectors and checking the rank of the 9×9 matrix that is formed from them.

3. Find the reachable set from $\dot{x}(0) = 0$, $x(0) = 1$ for the control system

$$\ddot{x}(t) + (1 + u(t))x^3(t) = 0.$$

First, determine the evolution of the unforced system (the drift component of the flow). The unforced equations are given by

$$\ddot{x} = -x^3$$

which can be integrated to give

$$\frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 = c$$

where $c \in \mathbb{R}$. If one considers this equation in phase space, it becomes clear that the solutions satisfying this equation are periodic and symmetric about the origin.

Now consider the effect of the control. In state space form, the system equations are

$$\dot{z} = \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -x^3 \end{bmatrix} + \begin{bmatrix} 0 \\ -x^3 \end{bmatrix} u = f(z) + g(z)u$$

Considering the vector field g we see that we can change the velocity $\dot{x}$ by an arbitrary amount (positive or negative) at any point except along the axis $x_3 = 0$. Changing the value of $\dot{x}_3$ allows us to change which elipse the system is following. Note that any elipse can be chosen except the one corresponding to the origin. span $\mathbb{R}^2$ everywhere but at the origin. Therefore, from every point but the origin the set reachable with the flow from the ideal is an open set in $\mathbb{R}^2$.

By allowing the system to flow along the drift for a long enough time, the system will return to within any ϵ of the starting point $[1, 0]$ and using the flow. By switching the level sets using the control, any point in $\mathbb{R}^2 - \{0\}$ is reachable.

4. Do the same for the Euler rigid body equations with two applied torques

$$\begin{aligned}
\dot{\omega}_1 &= ((I_2 - I_3)/I_1)\omega_2\omega_3 + u_1 \\
\dot{\omega}_2 &= ((I_3 - I_1)/I_2)\omega_1\omega_3 + u_2 \\
\dot{\omega}_3 &= ((I_1 - I_2)/I_3)\omega_1\omega_2
\end{aligned}$$

Here we have

$$f_0 = \begin{bmatrix} ((I_2 - I_3)/I_1)\omega_2\omega_3 \\ ((I_3 - I_1)/I_2)\omega_1\omega_3 \\ ((I_1 - I_2)/I_3)\omega_1\omega_2 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Operating under only the drift, one can see that again this system is periodic, and that its motions live on S^3. Using a similar analysis to problem 3, one can construct methods of moving between level sets corresponding to the controls turned off. We see then that the reachable set is $\mathbb{R}^3$.

5. Find the submanifold of $\mathbb{R}^4$ that contains the point $(w, x, y, z) = (1, 0, 0, 0)$ and has as its tangent space the distribution spanned by

$$\begin{array}{lll} F_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, & F_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, & F_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ F_4 = w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w}, & F_5 = w\frac{\partial}{\partial y} - y\frac{\partial}{\partial w}, & F_6 = w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w} \end{array}$$

Rewriting these vector fields we have

$$F_1 = \begin{bmatrix} 0 \\ -y \\ x \\ 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 \\ z \\ 0 \\ -x \end{bmatrix}, \ F_3 = \begin{bmatrix} 0 \\ 0 \\ z \\ -y \end{bmatrix}, \ F_4 = \begin{bmatrix} -x \\ w \\ 0 \\ 0 \end{bmatrix}, \ F_5 = \begin{bmatrix} -y \\ 0 \\ w \\ 0 \end{bmatrix}, \ F_6 = \begin{bmatrix} -z \\ 0 \\ 0 \\ w \end{bmatrix}$$

The manifold $\{(x, y, z, w) | x^2 + y^2 + z^2 + w^2 = 1\}$ satisfies the required conditions. Note that to form the tangent space for a manifold, one method is to construct a normal to the surface, and use the relation $n(p_0) \cdot (p - p_0) = 0$–the dot product of a vector in the tangent space at p_0 and the normal at that point must be zero. For this system, the normal is $n(p_0) = d(x^2 + y^2 + z^2 + w^2 - 1)|_{p_0} = [2w_0, \ 2x_0, \ 2y_0, \ 2z_0] = [1, \ 0, \ 0, \ 0]$. Evaluating the relation defining the tangent vectors gives $w - w_0 = 1$, so the allowable tangent vectors at $[1, \ 0, \ 0, \ 0]$ are given by $[1, \ x, \ y, \ z]$. With three independent quantities, vector fields that span this space are $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[0, 0, 0, 1]$. Similar computations can be done at any point of the manifold and the above vector fields.

6. Find the Lie algebra generated by the two vector fields

$$F_1 = x^2\frac{\partial}{\partial z}, \ F_2 = (z + x)\frac{\partial}{\partial x}$$

The vector fields can be rewritten as

$$F_1 = \begin{bmatrix} 0 \\ 0 \\ x^2 \end{bmatrix}, \ F_2 = \begin{bmatrix} x + z \\ 0 \\ 0 \end{bmatrix}$$

The Lie brackets we can generate are

$$[F_1, F_2] = \begin{bmatrix} x^2 \\ 0 \\ -2x(x + z) \end{bmatrix}, \ [F_1, [F_1, F_2]] = \begin{bmatrix} 0 \\ 0 \\ -4x^3 \end{bmatrix}, \ [F_2, [F_1, F_2]] = \begin{bmatrix} x(3x + 4z) \\ 0 \\ -2(x + z)(2x + z) \end{bmatrix}$$

Continuing to take Lie brackets will show that the Lie algebra is infinite dimensional.