

AA599: Geometric Methods for Nonlinear Control Systems

Homework # 7

Due: Friday May 28, 5:00pm

All problems have equal value. Please show all work, not just final answers.

- (Khalil 3.3) For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

$$(1) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned} \quad (2) \quad \begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$(3) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_1 \end{aligned} \quad (4) \quad \begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

- (Khalil 3.18) Consider Lienard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

- Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.
 - Using $V(x) = -\int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.
 - Repeat (b) using $V(x) = \frac{1}{2}[x_2 + \int_0^{x_1} h(y)dy]^2 + \int_0^{x_1} g(y)dy$.
- (Khalil 3.24) Consider the Lyapunov equation $PA + A^T P = -C^T C$, where the pair (A, C) is observable. Show that A is Hurwitz if and only if there exists $P = P^T > 0$ that satisfies the equation. Furthermore, show that if A is Hurwitz, the Lyapunov equation will have a unique solution.
Hint: Apply LaSalle's theorem and recall that for an observable pair (A, C) , the vector $C \exp(At)x \equiv 0 \forall t$ if and only if $x = 0$.
 - (Khalil 3.25) Consider the linear system $\dot{x} = (A - BR^{-1}B^T P)x$, where (A, B) is controllable, $P = P^T > 0$ satisfies the Riccati equation

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

$R = R^T > 0$, and $Q = Q^T \geq 0$. Using $V(x) = x^T P x$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable when

- $Q > 0$
 - $Q = C^T C$ and (A, C) is observable; see hint of the previous exercise.
- Consider the control system

$$\dot{x}(t) = \sum_{i=1}^m g_i(x(t))u_i(t)$$

with $x(t)$ taking on values in a differentiable manifold X . Assume that the system is controllable. Show that the control u that drives the system from the state x_0 to the state x_1 in one unit of time and minimizes

$$\eta = \int_0^1 \sum_{i=1}^m u_i^2 dt$$

is such that $\|u(t)\|$ is constant on the interval $[0, 1]$.