AA599: Geometric Methods for Nonlinear Control Systems
Homework #2
Due: Tuesday April 13, 5:00pm

All problems have equal value.

1. Find the distribution generated by the vector fields in $\mathbb{R}^3$

$$F_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ F_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

   Find a manifold in $\mathbb{R}^3$ that includes the point $(1,0,0) = (x_0, y_0, z_0)$ and has a tangent space spanned by these vector fields. (Please be careful about singular points if there are any.)

2. Find the matrix Lie algebra generated by A and B

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   You are free to use Mathematica or a similar program if helpful.

3. Find the reachable set from $\dot{x}(0) = 0$, $x(0) = 1$ for the control system

$$\ddot{x}(t) + (1 + u(t))x^3(t) = 0.$$

4. Do the same for the Euler rigid body equations with two applied torques

$$\begin{aligned} \dot{\omega}_1 &= ((I_2 - I_3)/I_1)\omega_2\omega_3 + u_1 \\ \dot{\omega}_2 &= ((I_3 - I_1)/I_2)\omega_1\omega_3 + u_2 \\ \dot{\omega}_3 &= ((I_1 - I_2)/I_3)\omega_1\omega_2 \end{aligned}$$

5. Find the submanifold of $\mathbb{R}^4$ that contains the point $(w, x, y, z) = (1, 0, 0, 0)$ and has as its tangent space the distribution spanned by

$$\begin{array}{lll} F_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, & F_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, & F_3 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ F_4 = w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w}, & F_5 = w\frac{\partial}{\partial y} - y\frac{\partial}{\partial w}, & F_6 = w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w} \end{array}$$

6. Find the Lie algebra generated by the two vector fields

$$F_1 = x^2\frac{\partial}{\partial z},\ F_2(z + x)\frac{\partial}{\partial x}$$