AA599: Geometric Methods for Nonlinear Control Systems
Homework #1
Due: Tuesday April 6, 5:00pm

All problems have equal value.

1. The axioms of a Lie algebra demand that $[a, b] = -[b, a]$ and that the Jacobi identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ be satisfied. Suppose that we wish to construct a Lie algebra from generators, imposing no additional conditions beyond these two. To be specific, suppose we start with two elements which we label x and y. We can take one bracket and get $[x, y]$. The other "bilinear" term $[y, x]$ is clearly linearly dependent on $[x, y]$. There are several trilinear terms $[x, [x, y]]$, $[y, [y, x]]$, etc. How many linearly independent trilinear terms are there? How many linearly independent quadralinear terms are there?

2. Write the equations of motion for a point mass constrained to a sphere, but otherwise free of external forces. Be explicit about the singularities of the coordinate systems you use.

3. Consider the set of all strictly proper rational functions with real coefficients and monic denominators of degree 2 and without common factors–objects of the form

$$g(s) = \frac{as + b}{s^2 + cs + d}$$

with (a, b, c, d) all real but subject to the common factor condition. Show that this set is the disjoint union of three connected manifolds, each of dimension four.

4. Express the kinetic energy of a rigid body in terms of the matrix

$$V = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & v_1 \\ -\omega_3 & 0 & \omega_1 & v_2 \\ \omega_2 & -\omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where ω is the angular velocities about a set of principal axes and v is the velocity of the center of mass. You will, of course, need to introduce a matrix representing the mass properties of the rigid body.

5. Derive the conditions given in Isidori, equations 1.8, 1.9, 1.11, 1.12.