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Axisymmetric flowing equilibriaof a two-fluid plasma

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1. Introduction

It has been recognized that shear flow may have a strong effect on confinement and stability. Recent Spherical Torus (ST) [Ref.1] and Conpact Toroid (CT) [Ref.2] experiments have observed the flow. For these ST and CT, it is not apparent whether the single-fluid model is adequate because these configurations (especially CT) have relatively small size parameter S_*:

$$S_* \equiv r_s/\ell_i = r_s\omega_{pi}/c\,. \tag{1.1}$$

and the single-fluid model assumes $S_* \to \infty$. Here r_s is the radius of the outer boundary of the plasma. To investigate the effect of the flow on the high beta plasmas, we have developed the formalism of equilibrium and stability analyses of a flowing two-fluid plasma [Ref.3]. However, the results shown were equilibria with only fairly large flow ($\sim V_A$). To compare the stability property of the static equilibrium with that of the flowing equilibrium, equilibriua with an arbitrary magnitude of flow are necessary. Hence, we have developed an improved formalism of *equilibrium analysis* of a flowing two-fluid plasma. In Sec.2, the basic equations for the present two-fluid model are shown. Introducing the generalized vorticities of each species [Ref.4] reduces the equations of motion to a compact form. In Sec.3, the formalism of the equilibrium analysis is shown. Two-fluid axisymmetric equilibria are described by coupled equations for the generalized vorticity surfaces of each species. In Sec.4, 2-D equilibria of ST and CT computed numerically are shown where purely azimuthal ion flow is assumed. In addition, an analytic equilibrium is also shown. Two-fluid effects are measured for these equilibria. The azimuthal ion flow and the beta value as well as the size parameter are found to play important roles in two-fluid effects. Sec.5 summarizes the paper and present a plan for future work.

2. Ideal two-fluid dynamical system

The plasma consists of ion and electron fluids. The density n is assumed to be uniform, constant in time, and the same for the both species (quasineutrality). Then the equations of motion, continuity, Ampere's, Faraday's, and Gauss's magnetism laws are

$$\partial\mathbf{P}_\alpha/\partial t - \mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha = -\nabla H_\alpha; \qquad \nabla\cdot\mathbf{u}_\alpha = 0 \tag{2.1,2.2}$$

$$\nabla\times\mathbf{B} = \frac{4\pi en}{c}\left(\mathbf{u}_i - \mathbf{u}_e\right); \qquad \nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}; \qquad \nabla\cdot\mathbf{B} = 0 \tag{2.3,2.4,2.5}$$

respectively. Here, $\mathbf{u}_\alpha$ and p_α are the fluid velocity and pressure of each species ($\alpha = i, e$) and n is the density. **E**, **B** are the electric and magnetic fields. Introducing the canonical momenta $\mathbf{P}_\alpha = m_\alpha\mathbf{u}_\alpha + q_\alpha\mathbf{A}/c$ and the generalized vorticities $\mathbf{\Omega}_\alpha = \nabla\times\mathbf{P}_\alpha = m_\alpha\nabla\times\mathbf{u}_\alpha + (q_\alpha/c)\mathbf{B}$ of the both species [Ref.4], the compact form of equations of motion, i.e. Eq.(2.1) is derived where m_α and q_α are the particle mass and charge. Here we restrict attention to a hydrogen plasma $q_i = e$, $q_e = -e$. **A** is the vector potential of the magnetic field. The *total* enthalpy H_α is the sum of thermodynamic (enthalpy), flow energy and electrostatic energy parts

$$H_\alpha \equiv \frac{p_\alpha}{n} + \frac{1}{2} m_\alpha u_\alpha^2 + q_\alpha \phi_E . \quad (2.6)$$

where ϕ_E is the scalar potential of the electric field.

Hereafter the electron mass is usually neglected since our interest is in low-frequency stability. This is consistent with the quasineutrality assumption and the neglect of the displacement current in Eq. (2.3). In the following section, we restrict attention to time-independent axisymmetic equilibrium quantities, e.g. $\mathbf{u}_\alpha = \mathbf{u}_\alpha(r,z)$ where the cylindrical coordinate system (r,θ,z) is adopted.

3. Equilibrium equations

3.1 Coupled equations which describe axisymmetric two-fluid equilibria:

The time-independent equilibrium motion equations (2.1) can be expressed as

$$\mathbf{u}_\alpha \times \mathbf{\Omega}_\alpha = \nabla H_\alpha \quad (3.1)$$

For axisymmetric equilibrium, the magnetic flux function $\psi(r,z)$ and the flow stream functions $\psi_\alpha(r,z)$ are useful to express the poloidal components of the divergence-free magnetic field and the species flow velocities respectively:

$$\mathbf{B}_p(r,z) = \nabla\psi \times \hat{\boldsymbol{\theta}}/r , \qquad \mathbf{u}_{\alpha p}(r,z) = \nabla\psi_\alpha \times \hat{\boldsymbol{\theta}}/nr \quad (3.2,3.3)$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction. Further we introduce the *generalized stream functions* $\Psi_\alpha(r,z)$ to express the poloidal (r,z) parts of the generalized vorticities of each species:

$$\mathbf{\Omega}_{\alpha p}(r,z) = (q_\alpha/c)\nabla\Psi_\alpha \times \hat{\boldsymbol{\theta}}/r \quad (3.4)$$

Since $\mathbf{\Omega}_{\alpha p} = m_\alpha(\nabla\times\mathbf{u}_\alpha)_p + (q_\alpha/c)\mathbf{B}_p$, the following relations hold:

$$\Psi_i = \psi + (m_i c/en) r u_{i\theta} \qquad \Psi_e = \psi \quad (3.5), (3.6)$$

Thus the *azimuthal* ion flow causes the difference between Ψ_i and ψ.

From the equations of motion, two relations follow at once: $\mathbf{\Omega}_\alpha \cdot \nabla H_\alpha = 0$ and $\mathbf{u}_\alpha \cdot \nabla H_\alpha = 0$. Then we have

$$\frac{p_\alpha}{n} + \frac{1}{2} m_\alpha u_\alpha^2 + q_\alpha \phi_E = H_\alpha(\Psi_\alpha) \quad (3.7)$$

$$\psi_\alpha = \psi_\alpha(\Psi_\alpha) \quad (3.8)$$

Thus the total enthalpies H_i, H_e and the species stream functions ψ_i, ψ_e are arbitrary *surface functions* of their respective surface variables Ψ_α. Using these surface functions, the equilibrium motion equation (3.1) for each species has only a component parallel to $\nabla\Psi_\alpha$, and arrives at the following coupled equations for Ψ_i and $\psi(=\Psi_e)$:

$$(4\pi m_i/n) r^2 (d\psi_i/d\Psi_i)\nabla\cdot\left[(d\psi_i/d\Psi_i)\nabla\Psi_i/r^2\right] =$$
$$= (4\pi e/c)^2(\psi_i - \psi_e)(d\psi_i/d\Psi_i) + (\psi - \Psi_i)/\ell_i^2 + 4\pi n r^2\, dH_{i0}/d\Psi_i \quad (3.9)$$

$$\Delta^*\psi = (4\pi e/c)^2(\psi_i - \psi_e)(d\psi_e/d\psi) + (\psi - \Psi_i)/\ell_i^2 - 4\pi n r^2\, dH_{e0}/d\psi \quad (3.10)$$

where the poloidal and toroidal parts of the Ampere's law, $rB_\theta = (4\pi/c)\sum_\alpha q_\alpha\psi_\alpha$ and $\Delta^*\psi = -(4\pi/c)\sum_\alpha q_\alpha r u_{\alpha\theta}$ are used to eliminate the toroidal components of the magnetic field and the flow of each species. Two-fluid equilibria are described by the coupled equations Eqs.(3.9) and (3.10). These are a reduced form of the more general system developed earlier [Eqs. (29) and (30) of Ref.5].

The pressure can be found *a posteriori* using Eq. (3.7). These can be summed to give the Bernoulli equation in terms of the total pressure, $p = p_i + p_e$:

$$p/n + m_i u_i^2/2 \quad = \quad H_i + H_e \tag{3.11}$$

3.2 Comparison between the single- and the two-fluid equilibrium models:

To see the difference between the single- and two-fluid equilibrium models, we introduce the alternative forms of the equations of motion:

$$m_i \mathbf{u}_i \bullet \nabla \mathbf{u}_i = -\nabla(p_i + p_e)/n + (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi \tag{3.12}$$

$$\mathbf{E} + \mathbf{u}_i \times \mathbf{B}/c + \mathbf{F}_{2F} = 0 \tag{3.13}$$

The Eqs.(3.12) and (3.13) are equivalent to Eqs.(3.1) for the species. The difference between the single- and the two-fluid equilibrium models is $\mathbf{F}_{2F}$:

$$\mathbf{F}_{2F} = [\nabla p_e - (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi]/(en). \tag{3.14a}$$

Using Eq.(3.12), this two-fluid correction can be expressed in another form:

$$\mathbf{F}_{2F} \equiv -(1/e)\left[\nabla\left(p_i/n + m_i u_i^2/2\right) - m_i \mathbf{u}_i \times (\nabla \times \mathbf{u}_i)\right] \tag{3.14b}$$

While $\mathbf{F}_{2F}$ represents the electron pressure gradient and the Hall effects [Eq.(3.14a)], it can also be viewed as reflecting the ion pressure gradient and inertia effects [Eq.(3.14b)]. When the following condition holds,

$$|\mathbf{u}_i \times \mathbf{B}/c| \approx |\mathbf{E}| \gg |\mathbf{F}_{2F}|, \tag{3.15}$$

the single-fluid model is adequate. Otherwise two-fluid effects are important. Two important factors are related to the size of $\mathbf{F}_{2F}$. (1) For the single-fluid model, the ion flow and the magnetic field share a single family of characteristic surfaces, ψ = *const*. By contrast, the two-fluid model has *two* families of characteristic surfaces: Ψ_i = *const*; $\Psi_e(=\psi)$ = *const*. This difference springs from the self-consistent treatment of ion inertia, i.e. the second term in Eq. (3.14b). (2) Ion flow perpendicular to the magnetic field springs only from the $\mathbf{E} \times \mathbf{B}$ drift in the single-fluid model. Missing are correction terms appearing in $\mathbf{F}_{2F}$: the diamagnetic and inertial ion drifts.

4. Results

Here we will find several equilibria in a toroidal geometry. For these equilibria, the average beta is calculated:

$$<\beta>_M \quad \equiv \quad <nT>_M / <nT + B^2/8\pi>_M \tag{4.1}$$

The average is on the symmetry plane ($z = 0$). To measure the global two-fluid effects, we introduce a two-fluid index:

$$f_{2F} = <|\mathbf{F}_{2F} \times \mathbf{B}|>_M / <|\mathbf{E} \times \mathbf{B}|>_M \tag{4.2}$$

If $f_{2F} \geq 1$ ($f_{2F} \ll 1$), the two fluid effect is significant (negligible). The usefulness of this index is verified by the analytic solution which is obtained in the following subsection. In the remainder of this section, we adopt a dimensionless variable scheme based on the reference length r_s and the reference magnetic field B_R (value of B_z at $r = r_s$, $z = 0$). The size parameter is $S_* = r_s/\ell_i$ as in Eq.(1.1).

Consider the special case of purely azimuthal ion flow. This implies a particular choice of one of the arbitrary functions;

$$\psi_i(\Psi_i) \quad = \quad 0 \tag{4.3}$$

For the remaining arbitrary functions, we consider a particular example:

$$dH_i/d\Psi_i = C_{Hi0} + C_{Hi1}\Psi_i, \tag{4.4}$$

$$dH_e/d\psi = C_{He0} + C_{He1}\psi + C_{He3}\psi^3 \tag{4.5}$$

$$\psi_e(\psi) = C_{BT} + C_{\psi e1}\psi + C_{\psi e2}\psi^2 \tag{4.6}$$

where the various *C*'s are constant parameters. Relationship between the two surface variables follows from Eq. (3.9):

$$\Psi_i = \frac{\psi + (C_{Hi0}/S_*^2)r^2}{1-(C_{Hi1}/S_*^2)r^2} \tag{4.7}$$

Substituting this into Eq. (3.10) gives the differential equation for ψ,

$$\Delta^*\psi + \left\{S_*{}^2\left(C_{\psi e1}{}^2 + 2C_{\psi e2}C_{BT}\right) + r^2\left[C_{He1} + \frac{C_{Hi1}}{1-r^2(C_{Hi1}/S_*{}^2)}\right]\right\}\psi +$$

$$+3S_*^2C_{\psi e1}C_{\psi e2}\psi^2 + \left(2S_*^2C_{\psi e2}^2 + r^2C_{He3}\right)\psi^3 + S_*^2C_{BT}C_{\psi e1} + r^2\left[C_{He0} + \frac{C_{Hi0}}{1-r^2(C_{Hi1}/S_*{}^2)}\right] = 0 \tag{4.8}$$

Since ψ has an arbitrary additive constant, the boundary condition $\psi = 0$ is imposed. After solving Eq.(4.8), the temperatures and the electrostatic potential are determined from Eqs.(3.7) for $\alpha = i$ and e assuming a constant temperature ratio,

$$T_i/T_e = \gamma_T = const \tag{4.9}$$

4.1 Analytic equilibrium:

Vanishing all of coefficients, C_{Hi1}, C_{He1}, C_{He3}, C_{BT}, $C_{\psi e1}$, and $C_{\psi e2}$, Eq. (4.8) reduces to

$$\Delta^*\Psi_e + \left(C_{He0} + C_{Hi0}\right)r^2 = 0 \tag{4.10}$$

This equation is the same as for the familiar Hill's vortex equilibrium. The magnetic flux function satisfying the elliptic boundary with elongation *E* is given by

$$\psi = -\left(r^2/2\right)\left(1 - r^2 - z^2/E^2\right) \tag{4.11}$$

where $C_{He0} + C_{Hi0} = -(4 + 1/E^2)$. Although ψ is the same as the Hill's vortex solution, this two-fluid equilibrium has ion flow and electric field:

$$\mathbf{u}_i = \hat{\boldsymbol{\theta}}r\left(C_{Hi0}/S_*\right), \qquad \mathbf{u}_e = -\hat{\boldsymbol{\theta}}r\left(C_{He0}/S_*\right) \qquad \text{(4.12), (4.13)}$$

$$\phi_E = \left[S_*u_{iE} + \gamma_T\left(4 + 1/E^2\right)/(\gamma_T + 1)\right]\psi + \left[u_{iE}^2/(\gamma_T + 1)\right]\left(r^2/2\right) \tag{4.14}$$

$$T_i = \gamma_T T_e = \left[\gamma_T/(\gamma_T + 1)\right]\left[-\left(4 + 1/E^2\right)\psi + u_{iE}^2 r^2/2\right] \tag{4.15}$$

where $u_{iE} = u_{i\theta}(r = 1) = C_{Hi0}/S_*$ .Because there is no toroidal magnetic field, this equilibrium has high averaged beta. If $4E^2 >> 1$, $<\beta>_M = \left(5 + 3u_{iE}^2\right)/\left(7 + 3u_{iE}^2\right) > 0.7$.

4.2 Numerical equilibria:

Figs. 1 and 2 represents the 2-D ST and CT equilibria. Fig.1a (Fig.2a) shows the contours of the magnetic flux function ψ/ψ_0 =0.0, 0.1, 0.3, 0.5, 0.7, 0.9 for the ST (CT) equilibrium, where ψ_0 is a value of ψ at the magnetic axis. At the diamond symbols on Fig.1a, the normal curvature $\kappa_n \equiv \nabla\psi \bullet \left[(\mathbf{b}\bullet\nabla)\mathbf{b}\right]/|\nabla\psi|$ vanishes where $\mathbf{b} \equiv \mathbf{B}/|\mathbf{B}|$. The inboard region between the symbols has "good curvature". Figs.1b-e (Figs.2b-e) show the profiles of the magnetic field, azimuthal ion flow, temperatures, and electric field on the symmetry plane for the respective equilibrium. The structures of magnetic field and the peak values of the azimuthal ion flow velocities and the temperatures of the ST and CT equilibria are relevant to the current NSTX [Ref.1] and TS-3 [Ref.2] experiments respectively. The beta value and the two-fluid index of the ST (CT) equilibrium are $<\beta>_M = 1.5\times10^{-2}$ ($<\beta>_M = 68\times10^{-2}$) and $f_{2F} = 0.052$ ($f_{2F} = 1.38$), respectively. Hence the two-fluid effects are negligible (significant) for the ST (CT) equilibrium.

4.3 Measurement of the two-fluid effects:

To illustrate the global two-fluid effect f_{2F}, it is calculated for the 1-D ST and CT equilibria with various values of flow verocity and average beta. The result is shown in Fig.3. The horizontal axis shows the average beta value. The vertical axis Γ represents the maximum (minimum) value of $u_{i\theta}$ when $u_{i\theta}$ is positive (negative). The circles (filled dots) on the Fig.3a (Fig.3b) represent individual ST (CT) equilibria. The number with each symbol represents its value of f_{2F}. Equilibria in the non-shaded (shaded) region have values of f_{2F} larger (smaller) than 0.8. The global two-fluid effect is significant in the equilibria located in the non-shaded region. Fig. 3 illustrates three important conclusions. The first is that, as the average beta increases, the region where the global two-fluid effect is significant becomes much broader. The second is that for a given average beta, for example $\langle\beta_T\rangle_M = 0.2$ in Fig.3a, f_{2F} has a maximum for certain value of Γ which is close to the ion diamagnetic drift velocity. The third is that, as S_* decreases, the region where the global two-fluid effect is significant becomes much broader.

5. Summary

Equilibrium properties of a flowing two-fluid plasma were studied. Introducing the generalized vorticities of the ion and the electron fluids reduces the basic equations to compact forms. Coupled equations for the surface variables associated with the generalized vorticities of each species were derived. The present two-fluid model naturally describes the thermal pressures of each species and the electric field. Assuming that the equilibrium has purely azimuthal ion flow, the 2-D equilibria of ST and CT were computed numerically. In addition, an analytic equilibrium was also obtained. The equilibria obtained numerically were relevant to the current experiments. Although the magnetic flux function of the analytic equilibrium is the same as the Hill's vortex solution, this *two-fluid* equilibrium includes ion flow and the electric field. For these equilibria shown here, the poloidal ion flow was neglected, while it is allowed in the model. Future work will include the poloidal ion flow.

A comparison between the single- and the two-fluid models was carried out. The characteristic differences became apparent. To measure the intrinsic two-fluid effects quantitatively, we introduced the two-fluid index. The usefulness of this index was verified by the analytic solution. The application of this index suggests that the two-fluid effect is significant for equilibria with the following three characteristics: (2) smaller S_* (1) higher beta, and (3) ion flow velocity closer to the ion diamagnetic drift. This result implies the followings: (a) Two-fluid effects are very significant for FRC plasmas; (b) Two-fluid effect can be significant *locally* in the region where the effective S_* is small, such as transport barriers in H-mode [Ref.6]; (c) Although the two-fluid effect is not very important for the beta values of present ST experiments, the two-fluid model will be necessary for the future high-beta regime. The application to these problems in stability analysis as well as equilibrium analysis is left for future works.

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