# Basics of Two-Fluid Plasma Physics—a Brief Summary

Loren Steinhauer

The elementary building blocks for a multi-fluid are the canonical momentum $\mathbf{P}_\alpha = m_\alpha\mathbf{u}_\alpha + q_\alpha\mathbf{A}/c$, and the generalized vorticity $\boldsymbol{\Omega}_\alpha = \nabla\times\mathbf{P}_\alpha$ (or $\alpha$-vorticity) where $m_\alpha$, $\mathbf{u}_\alpha$, $q_\alpha$ are the species mass, flow velocity and charge, and $\alpha = i,e$ denotes the species, and $\mathbf{A}$ is the vector potential. The quadratic invariant of a species is the self helicity, or "$\alpha$-helicity," the "density" of which is $\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$ . These are generalizations of helicities that appear in a simple fluids and MHD. For zero electron mass the electron helicity reduces to the familiar magnetic helicity, an invariant in ideal MHD. The evolution of the $\alpha$-helicities is governed by the helicity transport equation, derived from Maxwell's equations and the equations of motion.

Each helicity transport equation has the form, $n_\alpha D_\alpha(\mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha/n_\alpha)/Dt = \nabla\cdot[(\ldots)\boldsymbol{\Omega}_\alpha]$ + *friction*, where $n_\alpha$ is the density. The generalized vorticity appearing in the divergence term implies the existence of a "local" $\alpha$-helicity associated with these lines, $K_\alpha = (c^2/8\pi q_\alpha{}^2)\int_C d\tau\, \mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$, where $C$ is the volume occupied by a bundle of $\alpha$-vortex lines. The constant factor gives $K_\alpha$ the convenient units of energy-length. The total derivative $D_\alpha/Dt$ implies that the local $\alpha$-helicity convects with its own species. If an $\alpha$-vortex line does not intersect the system boundary, then in the strictly ideal (frictionless) case, the associated $\alpha$-helicity is constant. There is a circulation theorem, $\Gamma_\alpha = \oint_C \mathbf{P}_\alpha\cdot d\mathbf{x} = const$, where $C$ is an $\alpha$-vortex line, and $d\mathbf{x}$ is a differential length vector along that line. Each species has its own set of $\alpha$-vortex lines, its own local $\alpha$-helicities, and its own circulation theorem.

In the realistic case with friction, visco-resistive instabilities drive reconnections that break individual $\alpha$-vortex lines and destroy their identity. This is a case of non-uniform convergence because even a minute amount of friction is enough to compromise the local $\alpha$-helicities. The only quantities immune to these topology altering events are the global $\alpha$-helicities, $K_\alpha = (c^2/8\pi q_\alpha{}^2)\int_V d\tau\, \mathbf{P}_\alpha\cdot\boldsymbol{\Omega}_\alpha$, where $V$ is the system volume. Even global invariants may not be *rugged* in the sense that they are *more* "invariant" than the organized energy form, *i.e*. the magnetofluid energy $W_{mf} = \int_V d\tau\, (\Sigma m_\alpha n_\alpha u_\alpha{}^2 + B^2/8\pi)$, composed of the flow energy and the magnetic energy (the sum is over species). The ruggedness of the global $\alpha$-helicities has been supported by three arguments. (1) *Selective decay*: $W_{mf}$ decays more rapidly than $K_\alpha$ in thin reconnection layers. Properly applied, this argument must account for limits on viscous friction coefficients for sharp gradients. (2) *Inverse cascade*: the fluctuation spectrum of $\tilde{W}_{mf}(k)$ and $\tilde{K}_\alpha(k)$ satisfy the necessary conditions for a cascade toward larger scale objects ($k$ is the wave number of the disturbance). (3) *Stability to resistive modes*: $K_\alpha$ is less affected than $W_{mf}$ by resistive modes. Each of these is the generalization of arguments previously applied to verify the ruggedness of the magnetic helicity in weakly-dissipative MHD.

A minimum energy state is found formally by minimizing $W_{mf}$ subject ot invariant $\alpha$-helicities, and (given axisymmetric system boundary) the global angular momentum, $L_\theta = \int d\tau\, r\Sigma m_\alpha n_\alpha u_{\alpha\theta}$. The variation with respect to $\delta\mathbf{u}_\alpha$ leads to the flow equations: $n_\alpha(\mathbf{u}_\alpha - \Omega r\hat{\theta}) = (\lambda_\alpha/\ell_c^2)\boldsymbol{\Omega}_\alpha$ where $\lambda_\alpha$, $\Omega$ are the Lagrange multipliers associated with invariant $\alpha$-helicities and angular momentum, and $\ell_c = c/\omega_{pi} = (m_i c^2/4\pi e^2)^2$ is the length scale. An entropy maximization procedure subject to invariant $K_\alpha$, $L_\theta$, and total energy ($W_{mf}$ + thermal) leads to the same equation. In addition, a global Bernoulli equation links the pressure to the flow by a relation that applies throughout the system volume. Note that an important feature of a two-fluid minimum energy state is the length scale $\ell_c$. A two-fluid may or may not relax to the minimum energy state depending on whether the fast mechanisms have been stabilized.

References: L.C. Steinhauer and A. Ishida, Phys. Rev. Lett. **79**, 3423 (1997); Phys. Plasmas **5**, 2609 (1998).